impact math

focus on the future

Patterning & Algebra Module

Ontario Ministry of Education and Training
Queen's Park, Mowat Block,
Toronto, Ontario
M7A 1L2

Impact Math is a professional development program to help teachers of Grades 7/8 implement the new Mathematics curriculum. The program was developed by the Impact Math team at the Ontario Institute for Studies in Education of the University of Toronto (OISE/UT). The development of this resource document was funded by the Ontario Ministry of Education and Training. This document reflects the views of the developers and not necessarily those of the Ministry.

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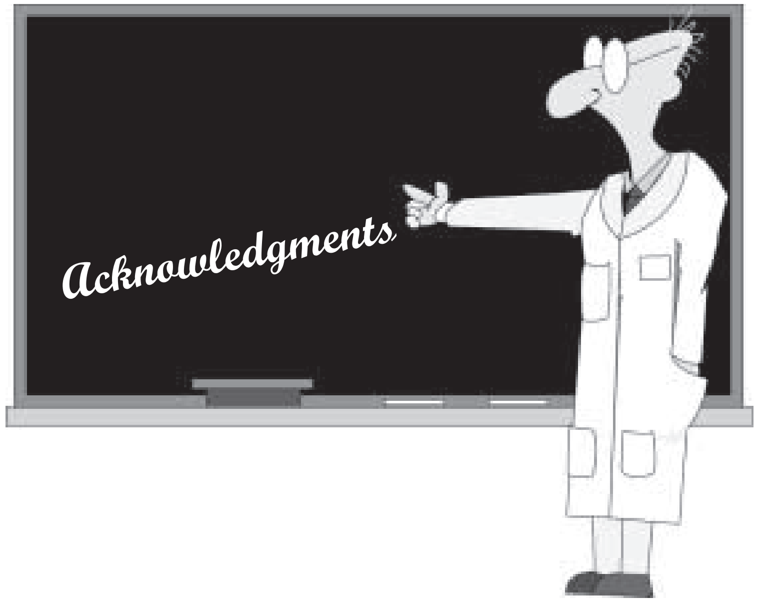

This module is the final product in a series of drafts, revisions and field tests conducted during the 1998–99 school year. Enhancing the usefulness of this booklet is the plethora of wonderful samples of student work that appear under the heading "WHAT YOU MIGHT SEE." For these samples we are deeply indebted to the Grade 7 and 8 students of the following teachers:

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We wish to thank these excellent teachers and their students for their invaluable contribution to this resource. Mike Wierzba and Barry Scully served as the liaisons who conducted the field tests, gathering information from students and teachers and providing insightful suggestions regarding the scoring guides, the activities, and the content of this module.

We acknowledge a significant debt to the following members of the *Impact Math* cadre of trainers who made important modifications to the scoring guides to facilitate their use by teachers and strengthen their use as aids to report card preparation (see pp. 94-95).

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The high level of professionalism and the generosity of spirit evident in the contributions of all our colleagues mentioned above has imbued us with a sense of optimism toward mathematics education in Ontario.

Contents

PART I – PHILOSOPHY & RATIONALE

PART II – WHAT'S NEW IN PATTERNING AND ALGEBRA?

"YES, I AGREE THAT MAN IS MASTER OF HIS OWN DESTINY, BUT SOMETIMES IT HELPS IF YOU PASS ALGEBRA."

PART III – PATTERNING IN GRADE 7

PART IV– ALGEBRA IN GRADE 8

Introduction to the Modules

The Ontario Curriculum, Grades 1–8: Mathematics, issued in 1997, has redefined the elementary school mathematics curriculum for Ontario. New expectations for student learning require the teaching of new mathematical topics as well as a shift in emphasis of content previously taught. In particular, the new document reflects the growing need for students to expand their skills in processing information, managing data, problem solving, and using technology to achieve these ends. While there is a reduced attention to rehearsing rote skills, such as long division with large divisors or extraction of roots by the formal method, there is a reaffirmation of the need for students to master the multiplication tables and fundamental pencil-and-paper skills that underpin arithmetic facility. Such skills are intended to support the intelligent use of technology in performing complex computations of the type that arise in so-called “real world” contexts.

Implicit in this document is the demand for new or revised methods of instruction and assessment. Educational research of the past twenty years has mounted a compelling argument for a knowledge-building approach to instruction (see page 9) that reduces the role of the teacher as purveyor of information and enhances the teacher’s role as facilitator of learning. With this shift in instructional methodology comes a corresponding demand for change in methods of assessment (see pages 10–12).

The call for such changes in curriculum, instruction, and assessment has created a need for teachers of Grades seven and eight to plan new programs in mathematics from the plethora of print and electronic resources currently available. Since most of these teachers are responsible for many subject areas in addition to mathematics, the consolidation of these materials into a set of coherent lessons is daunting. To support teachers in this quest, the Ministry of Education and Training has commissioned a set of five modules (of which this is one) that gather together many of the extant resources in a single reference package. Each module addresses one of the five strands in the new curriculum.

Though they address different content strands, all modules have the same format. Part I outlines the rationale underpinning the ideas and activities developed in the module. Part II provides a brief instruction for teachers on the new content or approaches in that strand. Part III provides a set of four sample activities for Grade 7. Together these constitute an authentic task designed to consolidate and extend earlier developmental activities. This unit is intended to model the instructional and assessment philosophies discussed in Part I. **It is not intended to cover the entire content of the strand, nor to replace any resources presently used, but rather to supplement the current program.** Included in Part III under the heading “What You Might See” are samples of student work, classified by achievement level, and presented opposite a rubric that will help you assess the work of your students. Part IV parallels Part III, except it is keyed to the Grade 8 unit. However, it is recommended that all teachers familiarize themselves with the contents of both Parts III and IV. Part IV concludes with a selected list of appropriate print and media resources at the Grade 7-8 levels and some useful Internet addresses to fulfill the intent that the module provide a single reference to help teachers implement the new curriculum.

The Rationale for Patterning & Algebra

Patterning

The rationale for the emphasis on patterning in the Ontario mathematics curriculum is given on page 52 of *The Ontario Curriculum, Grades 1–8: Mathematics:*

> *One of the central themes in mathematics is the study of patterns and functions. This study requires students to recognize, describe, and generalize patterns and to build mathematical models to predict the behaviour of real-world phenomena that exhibit observed patterns. Exploring patterns helps students develop both mathematical competence and an appreciation of the aesthetic qualities of mathematics.*

This rationale underlines the view that mathematical exploration is the search for pattern, and the means through which an understanding of mathematical concepts emerges. The National Council of Teachers of Mathematics (NCTM) provides further elaboration of this conception in *Curriculum and Evaluation Standards for School Mathematics: Addenda Series – Patterns and Functions* (p. 1):

> *In a very broad sense, the observation of patterns lies at the heart of acquiring understanding and knowledge in many disciplines such as science, history, economics, and social science. There is no area in which the study of patterns is as fundamental as it is in mathematics. Mathematicians observe patterns; they conjecture, test, discuss, verbalize, and generalize these patterns. Through this process they discover the salient features of the pattern, construct understandings of the concepts and relationships, develop a language to talk about pattern, integrate, and discriminate between the pattern and other patterns. When relationships between quantities in a pattern are studied, knowledge about important mathematical relationships and functions emerges.*

Algebra

The rationale for the inclusion of algebra in the Ontario mathematics curriculum is also given on page 52 of *The Ontario Curriculum, Grades 1–8: Mathematics:*

> *Algebra is the language through which most of mathematics is communicated. The focus of the study in Grades 7 and 8 is first on understanding how the language of algebra can be used to generalize a pattern or a relationship. A second focus is on using algebra as a problem-solving tool – a means of clarifying concepts at an abstract level before applying them.*

In the sample units presented in Parts III and IV of this document, you will observe how the development of algebraic concepts can emerge out of the search for pattern.

The Role of Technology in the Patterning & Algebra Strand

The policy on the use of technology, as embodied in *The Ontario Curriculum, Grades 1–8: Mathematics,* is stated on page 7 of that document:

> *Students are expected to use calculators or computers to perform operations that are lengthier or more complex than those covered by the pencil-and-paper expectations. When students use calculators and computers to perform operations, they are expected to apply their mental computation and estimation skills in predicting and checking answers. Students will also use calculators and computers in various experimental ways to explore number patterns and to extend problem solving.*

The rationale for this policy is clearly expressed on page 30 of the *National Council of Teachers of Mathematics 1997–98 Handbook:*

> *Technology is changing the ways in which mathematics is used and is driving the creation of new fields of mathematical study. Consequently, the content of mathematics programs and the methods by which mathematics is taught and learning assessed are changing. The ability of teachers to use the tools of technology to develop, enhance, and expand students' understanding of mathematics is crucial. These tools include computers, appropriate calculators (scientific, graphing, programmable, etc.), videodisks, CD-ROM, telecommunications networks by which to access and share real-time data, and other emerging educational technologies. Exploration of the perspectives these tools provide on a wide variety of topics is required by teachers.*
>
> *It is the position of the National Council of Teachers of Mathematics that the use of the tools of technology is integral to the learning and teaching of mathematics. Continual improvement is needed in mathematics curricula, instructional and assessment methods, access to hardware and software, and teacher education.*

The graphing calculator and the spreadsheet are two particularly useful tools in the teaching and learning of algebra. The search for a pattern in a sequence of numbers requires the discovery of a rule for calculating the general term of that sequence. When students conjecture an algebraic expression to represent the n^{th} term of a sequence, they can test their conjecture by entering the expression into a graphing calculator to create a table of values. By scrolling through the table, they can check whether their conjectured expression generates the given sequence. They can also obtain a graph of the algebraic expression and discover the relationship between the algebraic and geometric forms of a function. This approach is modelled in the unit on patterning in Part III of this document.

A spreadsheet defines the numerical value in each cell in terms of the values in other cells. This relationship is achieved by using expressions that are algebraic in form. The student activity on page 87 of this book invites students to define three different linear functions in a spreadsheet and use these to identify which of three different banquet hall rental plans offers the lowest cost. The activity then challenges students to alter the cost functions and determine how this changes the viability of each location. In this way, students not only learn how to create a spreadsheet, but they learn how it can be used to conduct “what if?” scenarios. In the field test, many students completed this activity successfully. A typical student response is presented at the bottom of page 92.

Understanding the Learning Process & Its Impact on Instruction

In this and the other four modules, we present activities that attempt to incorporate a range of instructional approaches. The students are sometimes given information and required to read, interpret, and apply it in an exercise. In other cases, the students must investigate, explore, and discover concepts that lurk beneath the surface of an activity. In some cases, the students will work individually, while in others they will work collaboratively or cooperatively. The activity in Part III of this module introduces students to the sequence of triangular numbers and challenges them to investigate this sequence to find a general rule for calculating the n^{th} triangular number. This investigation prompts them to search for a pattern, conjecture a formula, test their conjecture and report their findings. The activity in Part IV asks students to create tables and graphs to represent three different banquet hall rental agreements and to determine which of the three offers the lowest cost. Students are encouraged to work individually or in groups using graph paper, spreadsheets and (where available) graphing calculators as tools for exploration.

In view of these multiple perspectives on how children learn, one might assume that all traditional approaches to teaching will disappear as these philosophies are incorporated. However a response to the question "What should I see in a [NCTM] Standards-based mathematics classroom?" the *NCTM 1997–98 Handbook* presents a balanced and accessible image of effective instruction:

> *First and foremost, you'll see students doing mathematics. But you'll see more than just students completing worksheets. You'll see students interact with one another, use other resources along with textbooks, apply mathematics to real-world problems, and develop strategies to solve complex problems.*
>
> *Teachers still teach. The teacher will pose problems, ask questions that build on students' thinking, and encourage students to explore different solutions. The classroom will have various mathematical and technological tools (such as calculators, computers, and math manipulatives) available for students to use when appropriate. The teacher may move among the students to understand their thinking and how it is reflected in their work, often challenging them to engage in deeper mathematical thinking.*

Assessment: Rubrics & Achievement Levels

The changes in curriculum and instruction described on the preceeding pages have significant implications for assessment and evaluation. Among these implications is the shift from norm-referenced to criterion-referenced assessment, as described on page 1 of *The Assessment Standards for School Mathematics* published by the NCTM in 1995:

> *At present, a new approach to assessment is evolving in many schools and classrooms. Instead of assuming that the purpose of assessment is to rank students on a particular trait, the new approach assumes that high public expectations can be set that every student can strive for and achieve, that different performances can and will meet agreed-on expectations, and that teachers can be fair and consistent judges of diverse student performances.*

The Ontario Curriculum, Grades 1–8: Mathematics (see pp. 4–5) also embraces the move to criterion-referenced assessment and includes four levels of achievement for describing student performance:

> *High achievement is the goal for all students, and teachers, students, and parents need to work together to help students meet the expectations specified. The achievement levels are brief descriptions of four possible levels of student achievement. These descriptions, which are used along with more traditional indicators like letter grades and percentage marks, are among a number of tools that teachers will use to assess students' learning. The achievement levels for mathematics focus on four categories of skills: problem solving, understanding of concepts, application of mathematical procedures, and communication of required knowledge. When teachers use the achievement levels in reporting to parents and speaking with students, they can discuss with them what is required for students to achieve the expectations set for their grade.*

Descriptions of the four levels of achievement for problem solving, concepts, applications, and communication are shown on page 9 of that document. These are the levels for concept understanding:

knowledge/skills	Level 1	Level 2	Level 3	Level 4
Understanding of concepts	**The student shows understanding of concepts:**			
	– with assistance	– independently	– independently	– independently
	– by giving partially complete but inappropriate explanations	– by giving appropriate but incomplete explanations	– by giving both appropriate and complete explanations	– by giving both appropriate and complete explanations, and by showing that he or she can apply the concepts in a variety of contexts
	– using only a few of the required concepts	– using more than half the required concepts	–using most of the required concepts	– using all of the required concepts

A table such as the one above that describes levels of achievement is called a *rubric*. Included with the student activities, in this and the other modules, are rubrics and samples of student work that exemplify the levels of student performance as defined in *The Ontario Curriculum, Grades 1–8: Mathematics.*

Assessment: Rubrics & Achievement Levels

The release of the first module in this series, *Data Management & Probability*, was met with widespread enthusiasm. It confirmed our belief that teachers need and want materials to help them implement the new mathematics curriculum. Of particular interest to teachers are the issues associated with assessment and evaluation. The shift in emphasis from rote learning to higher-order processes, such as problem solving, drawing inferences, and communicating mathematical conclusions, requires that methods of performance assessment be added to the battery of devices that teachers use to assess mathematical learning. As observed in the NCTM publication *Curriculum and Evaluation Standards for School Mathematics: Addendum Series – A Core Curriculum* (1992):

> *Questions eliciting open-ended responses require more holistic approaches for scoring. Indirectly, they convey to students the need to communicate their ideas clearly and to construct their responses for a purpose. The impact on the curriculum of this type of assessment is to hold students accountable for demonstrating their understanding of connected ideas rather than displaying their proficiency with disconnected skills.* (p. 11)

One of the most important devices for the holistic scoring of higher-order tasks is the rubric. The rubric shown on page 10 is an example of what is called a "general rubric." In its publication *Assessment Standards for School Mathematics* (1995), the NCTM defines a general rubric as "an outline for creating task-specific rubrics" (p. 90). Furthermore it defines a "task-specific rubric" as a rubric that "describes levels of performance for a particular complex task and guides the scoring of that task consistent with relevant performance standards." In this module we present, under the heading WHAT YOU MIGHT SEE, samples of student responses to the activities. Large samples of student work collected during the field tests of these materials were used to create scoring guides. These guides are task-specific rubrics. You will notice however that they evaluate the "product," i.e., the student work, while the general rubric shown on page 10 includes an observational component of assessment (e.g., "with assistance," "independently"). Since there can be no observational component in the assessment of *completed* student work, the scoring guides in this book do not use phrases such as "independently" or "with assistance." **It is expected that teachers will use each scoring guide as a starting point in the development of a task-specific rubric that will evolve as it is used with students.**

On page 12, we offer some suggestions on how to develop task-specific scoring guides. However, it is important to recognize that the creation of rubrics is highly subjective and is more an art than a science. In the *TIMSS Monograph #1: Curriculum Frameworks for Mathematics and Science* (1993), Robitaille et al. issue this caveat:

> *Measuring educational achievement is difficult from both a conceptual and a practical perspective. What counts as "achievement" is not always easy to discern and even when a concept of achievement has been clearly explicated, ways and means for assessing it are not easily devised. The ongoing debate about educational measurement and the increasing number of alternative assessment approaches proposed in educational circles attest to this problem.* (p. 36)

Some Suggestions for Creating Your Own Scoring Guides

There are no set rules for constructing scoring guides. Each teacher will have personal preferences and individual conceptions that contribute a significant subjective component to this assessment instrument. Consequently there will be some variation among teachers in the levels of achievement assigned to a particular student response. However, the process described below presents the main elements in constructing scoring guides that many educators have found effective.

❶ If there are other teachers who are teaching mathematics at the same grade level, plan to set aside about 90 minutes to work together.

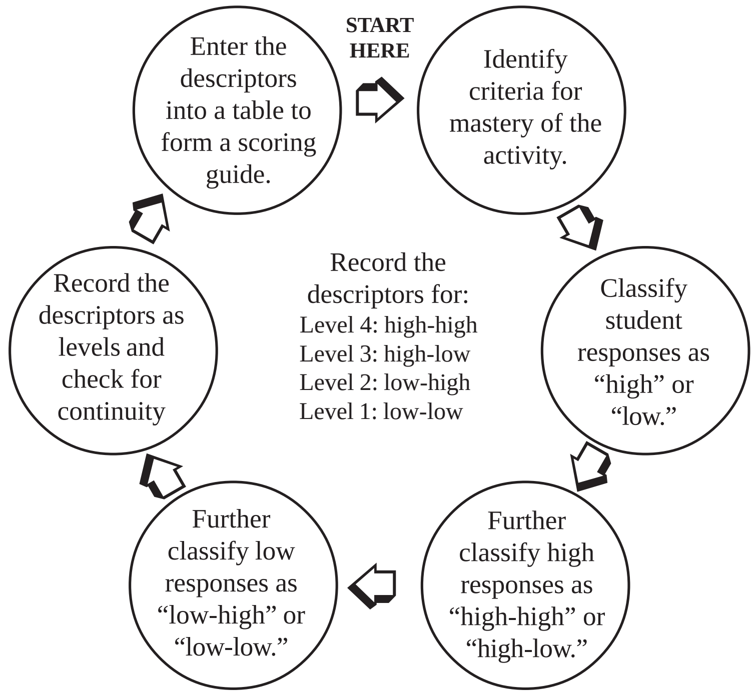
❷ Bring as many samples as possible of student work on the activity for which you are developing the scoring guide. Make enough copies so each of you has a complete set.

❸ Decide upon the kinds of responses that constitute mastery of the task. Identify responses that constitute various levels of partial mastery (about 20 minutes).

❹ Have each teacher individually assign the level "high" or "low" to each student response (about 20 minutes).

❺ When all student responses have been classified as "high" or "low," discuss and reach consensus on the rating of each response (about 20 minutes).

❻ Gather all the responses classified as "high" and repeat the process in ❹ and ❺, assigning the rating "high-high" or "high-low" to each of the "high" responses. Record descriptors used to determine each classification. Assign the "high-low" descriptors to "Level 3" and the "high-high" descriptors to "Level 4" (about 10 minutes).



Steps in Creating a Scoring Guide

❼ Repeat the procedure in ❻, classifying each of the "low" responses as "low-high" or "low-low." Reach consensus and record the descriptors. Assign the low-high descriptors to Level 2 and the low-low descriptors to Level 1 (about 10 minutes).

❽ Review the descriptors for the four levels to ensure that they form a continuum of increasing expectation and capture the criteria established in ❸. Record the descriptors in a scoring guide using a format such as in this book.

Helpful Resources for Creating Scoring Guides

Bryant, Deborah and Mark Driscoll. *Exploring Classroom Assessment in Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1998.

Danielson, Charlotte. *A Collection of Performance Tasks and Rubrics: Upper Elementary School Mathematics.* Larchmont, NY: Eye on Education, Inc., 1997.

Flewelling, Gary and Chuck Lemenchik. *Mathematics Assessment: Grades 7 & 8.* Toronto, Ontario: Gage Educational Publishing Company, 1997.

PART II

What's New in Patterning & Algebra?

Patterning as a Bridge to Algebra

New instructional approaches rather than new content represent the major change in the patterning and algebra strand. On page 7, we described briefly the rationale for the inclusion of patterning and algebra in the curriculum as stated in *The Ontario Curriculum Grades 1–8: Mathematics*. Here we provide a glimpse of how the teaching of patterns might be employed as a bridge into algebra.

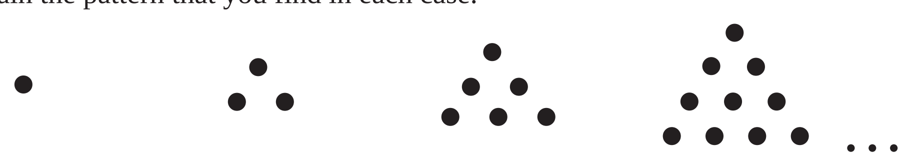
In its powerful publication, *Everybody Counts: A Report to the Nation on the Future of Mathematics Education* (1989), the Mathematical Sciences Education Board of the United States identified several important new visions for mathematics education that are shared by Canadian mathematics educators:

> *Mathematics is a living subject which seeks to understand patterns that permeate both the world around us and the mind within us. Although the language of mathematics is based on rules that must be learned, it is important for motivation that students move beyond rules to be able to express things in the language of mathematics. This transformation suggests changes in both curricular content and instructional style. It involves renewed effort to focus on:*
>
> - *Seeking solutions, not just memorizing procedures;*
> - *Exploring patterns, not just learning formulas;*
> - *Formulating conjectures, not just doing exercises.*
>
> *As teaching begins to reflect these emphases, students will have opportunities to study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized. They will be encouraged to see mathematics as a science, not as a canon, and to recognize that mathematics is really about patterns and not merely about numbers. (p. 84)*

It is expected that students in Ontario will study patterns informally through all the elementary grades. However, by Grade 7, an important expectation of students is that they can not only discover simple patterns in sequences, but also that they can generalize the pattern and describe how to calculate the general or n^{th} term of that sequence. At first, their recipe for finding the n^{th} term may be expressed in words, but gradually, as they learn the language of algebra, they will be able to write the n^{th} term as an algebraic expression. This is the heuristic and intuitive bridge to algebra afforded by the study of patterns.

To provide a transition from previous work with patterns, and to prepare students having difficulty for the work in the sample unit in Part III, two pattern "warm-up" templates are provided on pages 16 and 17. These warm-ups provide an informal approach to sequences such as the following:

> Look for a pattern in each sequence of diagrams and draw the one that comes next.
> Explain the pattern that you find in each case.



Students should be encouraged to use any manipulatives such as interlocking cubes or squared paper to explore these patterns. They should also be encouraged to explain orally how they would construct any particular term in the sequence. An excellent resource that provides activities for patterning with manipulatives leading up to the development of Pascal's triangle is *Teaching Mathematics with Manipulatives* by Mark A. Spikell, © 1993 by Allyn and Bacon.

The Changing Personality of a Variable

Traditionally, algebra has been taught in a formal way. Students have been introduced to the concept of a variable as an unknown – usually represented by the letter *x*. They have been taught the rules for identifying and collecting like terms, removing brackets, and applying the distributive law. Then the formal rules for manipulating algebraic expressions have been developed to solve a linear equation in the relentless quest to find the unknown value of *x*. A consequence of this formal approach has been that many students were left behind; soon to abandon the study of mathematics beyond grade 10. The new reform movement in mathematics issues the rallying cry, "algebra for all." The Mathematical Sciences Education Board in their publication *Everybody Counts* (cited on page 14 of this book) asserts:

> *The focus of school mathematics is shifting from a dualistic mission – minimal mathematics for the majority, advanced mathematics for a few – to a singular focus on a significant common core of mathematics for all students.* (p. 81)

To make algebra more accessible and to exploit the advantages of the new graphing technology, the formal approach has been abandoned in favour of a more exploratory student-centred approach. By exploring the pattern in a sequence and expressing in words and in symbols a method of calculating the n^{th} term of the sequence, the student discovers the meaning and the significance of algebraic notation. The variable *n* is then presented as a symbol that can take a variety of values, and the algebraic expression for the n^{th} term is presented as a function of *n*. For example, the sequence 1, 4, 9, 16, … runs through the values of n^2 as *n* runs through the positive integers: 1, 2, 3, 4, … As noted on page 52 of *The Ontario Curriculum Grades 1–8: Mathematics:*

> *In Grades 4 to 6, the focus of instruction shifts from exploring patterns to exploring functions. When students use graphs, data tables, expressions, equations, or verbal descriptions to represent a single relationship, they discover that different representations yield different interpretations of a situation. Through such activities, students learn informally that functions are things that can vary (variables) and that therefore have a changing relationship with other variables: changes in one variable result in changes in another.*

In the sample unit in Part III, students explore a variety of patterns including the properties of triangular numbers. In particular, they are presented with the problem of discovering an algebraic expression for the n^{th} triangular number. The four activities that comprise this unit offer guided instruction to help the students discover the expression in different ways. Spreading the development over several activities postpones some of the instructional guidance, thereby providing an opportunity for the more capable students to solve the problem without extra help. This Grade 7 unit lays the groundwork for the Grade 8 sample algebra unit presented in Part IV. In that unit, Jennifer and Steve's committee is faced with the task of organizing a fundraising party for which they must rent a banquet hall. Three different linear models are presented to the students in language form and the students must decide which of the three offers is the least expensive for various numbers of guests. To reach a decision, students must graph all three offers, represent them in tables, and construct the inequalities that define the numbers of guests for which each offer is optimal. In the process, students encounter linear functions of different varieties, see their graphs and the corresponding tables, and interpret their points of intersection. They come to understand the linear cost function as a measure of the changing cost as the variable *n* (number of guests) changes. In the field tests, students reported that they enjoyed the activity because they felt they were applying mathematics to a real problem. Their written reports (see pp. 90-92) reflect this satisfaction.

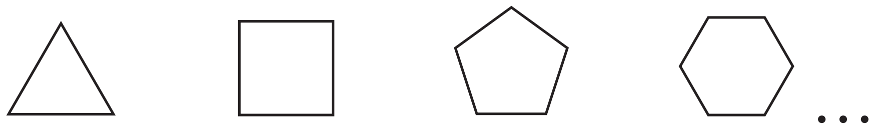
TEMPLATE

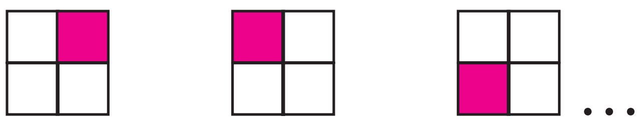
Pattern Sleuthing — Warm-up

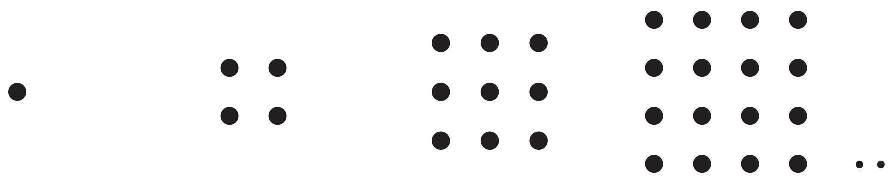
A Mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

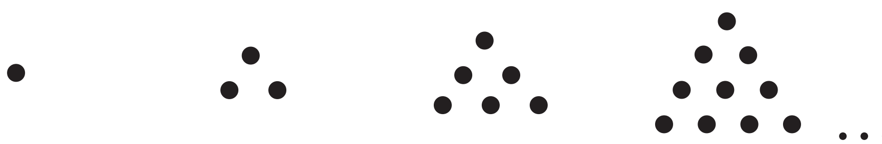
G. H. Hardy (number theorist)

Look for a pattern in each sequence of diagrams and draw the one that comes next. Explain the pattern that you find in each case.

❶ 

❷ 

❸ 

❹ 

Look for a pattern in each sequence. Describe the pattern that you discover. Then fill in the next three numbers.

❺ 7, 14, 21, 28, □, □, □, …

❻ 3, 7, 11, 15, □, □, □, …

❼ 3, 6, 10, 15, □, □, □, …

❽ 1, 4, 9, 16, □, □, □, …

❾ 3, 8, 15, 24, □, □, □, …

❿ 2, 4, 8, 16, □, □, □, …

⓫ 4, 6, 10, 18, □, □, □, …

⓬ 2, 6, 12, 20, □, □, □, …

Watch out for this one ☛ ⓭ 1, 1, 2, 3, 5, □, □, □, …

Write an algebraic expression for the n^{th} term for as many of the sequences from ❺ to ⓬ as you can.

Pattern Sleuthing — Warm-up

Complete each T-table. Describe in words the rule for calculating the output numbers.

❶

n	$4n$
1	4
2	8
3	12
4	16
5	
6	
7	

❷

n	$3n-2$
1	1
2	4
3	7
4	10
5	
6	
7	

❸

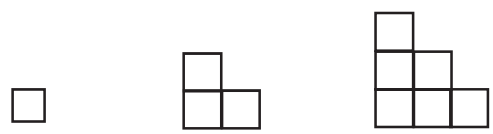
n	n^2
1	1
2	4
3	9
4	16
5	
6	
7	

❹

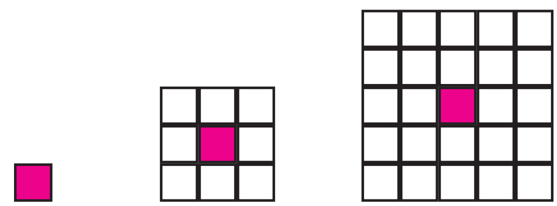
n	3^n
1	3
2	9
3	27
4	81
5	
6	
7	

Construct a T-table for $n = 1, 2, 3, \dots 7$ for each pattern. Use squared paper or interlocking cubes to check your answers.

❺



❻



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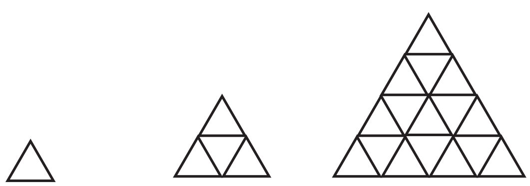
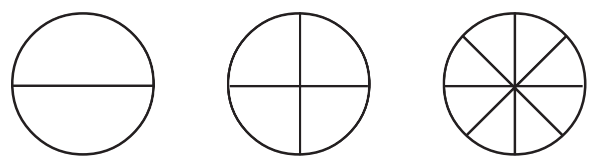


❽



Write an algebraic expression for the n^{th} output in each T-table.

Complete each T-table by filling in the missing value of n that corresponds to the given output.

❾

n	$9n$
1	9
2	18
3	27
•	•
•	•
•	•
☐	234

❿

n	$3n+5$
1	8
2	11
3	14
•	•
•	•
•	•
☐	56

⓫

n	n^2-3
1	-2
2	1
3	6
•	•
•	•
•	•
☐	438

⓬

n	2^n-1
1	1
2	3
3	7
•	•
•	•
•	•
☐	4095

Describe how you determined the missing value of n in each case.

If you have a graphing calculator or access to a spreadsheet, use the algebraic expressions in the T-tables above to create a table. Scroll down to check your answers.

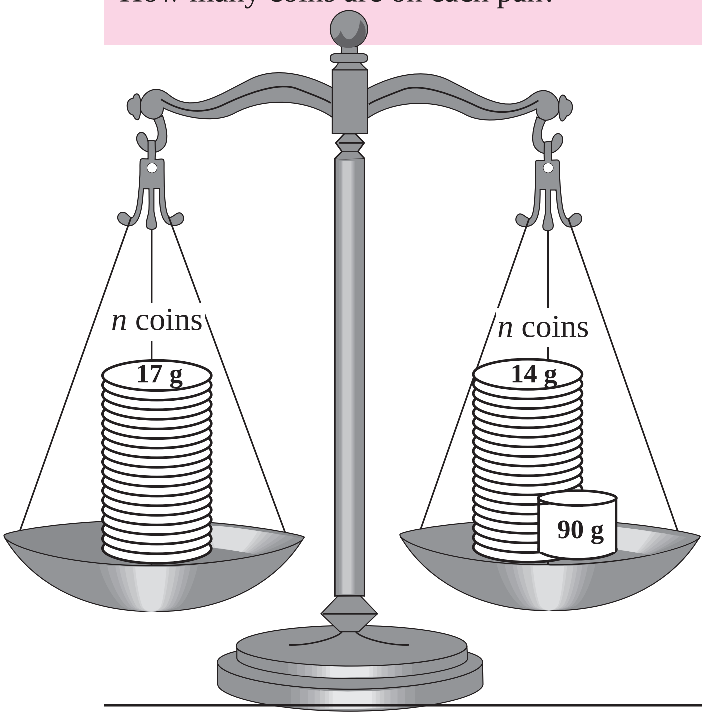
STUDENT READINESS ACTIVITY

Challenges in Reasoning

Problem 1

In Activities 2 and 3 (see pp. 71 & 79) you will solve these problems by graphing and creating a table. Can you solve them now using logical reasoning? If so, explain how.

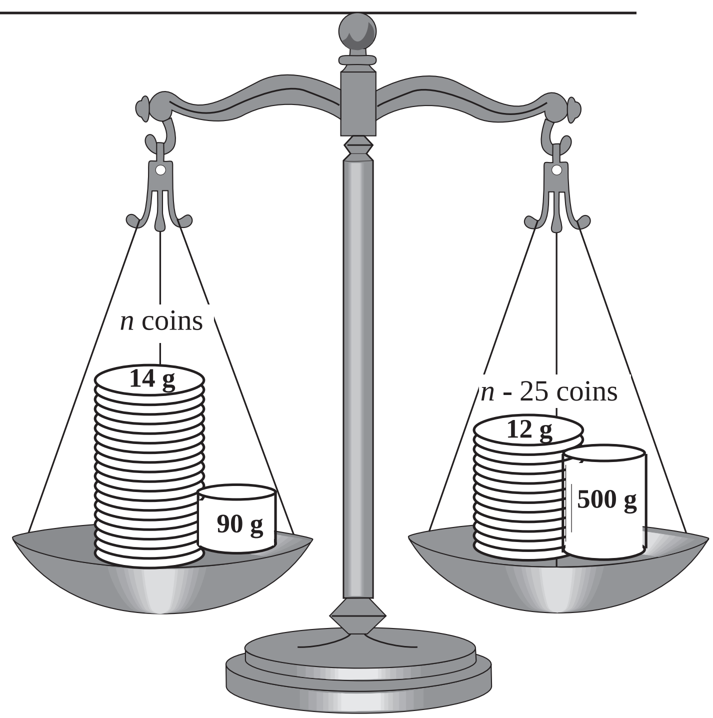
Mr. Scrooze is a miser who has coins with masses of 14 g and 17 g. He places a stack of 17-g coins on one pan of a two-pan balance and the same number of 14-g coins on the other pan. To balance both pans, he adds a 90-g mass to the pan with the 14-g coins. How many coins are on each pan?



See page 93 for the solution.

Problem 2

In his next balancing act, Mr. Scrooze has coins with masses of 12 g and 14 g. He places all his 14-g coins and his 90-g mass on one pan of a two-pan balance. He has 25 fewer 12-g coins which he places on the other pan. To balance both pans, he adds a 500-g mass to the pan with the 12-g coins. How many coins are on each pan?



See page 93 for the solution.

PART III

Patterning in Grade 7

Patterning & Algebra: Grade 7

Overall Expectations

By the end of Grade 7, students will:

• identify the relationships between whole numbers and variables;

• identify, extend, create, and discuss patterns using whole numbers and variables;

• identify, create, and solve simple algebraic equations;

• apply and discuss patterning strategies in problem solving situations.

Specific Expectations

(For convenient reference, the specific expectations are coded. For example, P&A 7-3 denotes the third Patterning and Algebra expectation in Grade 7.)

Students will:

Modelling

P&A 7-1 – describe patterns in a variety of sequences using the appropriate language and supporting materials;

P&A 7-2 – extend a pattern, complete a table, and write words to explain the pattern;

P&A 7-3 – recognize patterns and use them to make predictions;

P&A 7-4 – interpret a variable as a symbol that may be replaced by a given set of numbers;

P&A 7-5 – write statements to interpret simple formulas;

P&A 7-6 – present solutions to patterning problems and explain the thinking behind the solution process;

Linear Equations

P&A 7-7 – evaluate simple algebraic expressions by substituting natural numbers for the variables;

P&A 7-8 – translate simple statements into algebraic expressions or equations;

P&A 7-9 – solve equations of the form $ax = c$ and $ax + b = c$ by inspection and systematic trial, using whole numbers, with and without the use of a calculator;

P&A 7-10 – solve problems giving rise to first-degree equations with one variable by inspection or by systematic trial;

P&A 7-11 – establish that a solution to an equation makes the equation true (limit to equations with one variable).

Activity 1 – Teacher Edition

How Many Dots In A Triangular Array with n Dots on Each Side?

Expectations Addressed

P&A 7-1 describe patterns in a variety of sequences using the appropriate language and supporting materials.

P&A 7-2 extend a pattern, complete a table, and write words to explain the pattern.

P&A 7-3 recognize patterns and use them to make predictions.

Context

The recognition, extension, and description of patterns in sequences is developed in this unit in a multicultural context. Students are introduced to the idea that mathematics is the accumulation of intellectual contributions from many different societies and cultures. Beginning with the development of figurate numbers by the ancient Greeks in the sixth century B.C., the student is introduced to the idea that properties of numbers have fascinated humans since the beginning of recorded history. In Activity 1, students are encouraged to explore the sequence of triangular numbers, first with manipulatives and then using tables.

In Activity 2 (see p. 32), students encounter Pascal's triangle and learn of the contributions from the Chinese culture in the 14^{th} century. They fill in the missing numbers in the template for Pascal's triangle (see p. 56), and discover the diagonal line containing the sequence of triangular numbers. They are guided toward the discovery that the sum of the integers from 1 to n is the n^{th} triangular number.

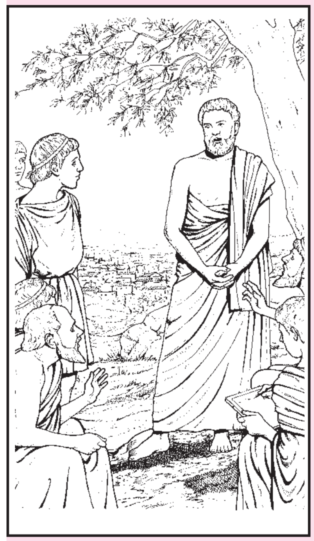
In Activity 3 (see p. 40), students learn that the Arabs and Hindus contributed to the mathematical development of algebra during the European dormancy of the Dark Ages. Students discover, by cutting squared paper, that the n^{th} triangular number is merely half the number of squares in a rectangle of n rows and n+1 columns. They are then guided toward expressing this relationship in algebraic notation. Thus they discover an algebraic formula for the n^{th} triangular number.

Activity 4 visits Gauss, the great 19^{th}-century German mathematician who incidently discovered how to sum the numbers from 1 to n at an early age. The students are prompted to use Gauss's method to find an expression for the sum of the numbers from 1 to n. This brings them to an alternative way to derive an algebraic formula for the n^{th} triangular number.

Activity 1 – Student Page

How Many Dots in a Triangular Array with n Dots on each Side?

Our earliest records from etchings in clay tablets to carvings in stone, reveal that humans through the ages have been fascinated with number patterns. The concept of number and the search for pattern has led to the development of the body of knowledge that we know today as *mathematics*. Contributions to mathematics have come from almost every culture and continent. In this unit, you will use number patterns to explore some number properties that have been investigated throughout history.

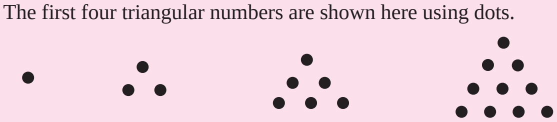


About 530 B.C. a group of philosophers and mathematicians in ancient Greece formed a secret society dedicated to the study of numbers and their properties. This secret society was known as the *Pythagoreans*, in honour of their distinguished leader, Pythagoras.

The Pythagoreans believed that the mysteries of the Universe could be explained in terms of number relationships. One of their members, Hippasus, proved that √2 cannot be represented as a common fraction. When he shared his discovery with people outside the Pythagorean cult, he was expelled for breaking his oath of secrecy. (Some accounts say he was killed by the Pythagoreans.)

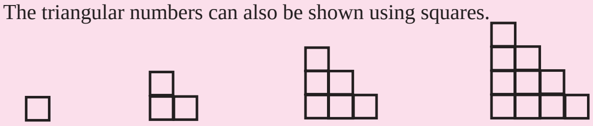
The Pythagoreans associated numbers with shapes. Numbers that can be represented by triangular arrays of dots (such that the n^{th} row contains n dots) were called *triangular numbers.* Numbers the can be displayed in n rows of n dots were called *square numbers,* and so on.

The first four triangular numbers are shown here using dots.



1 3 6 10

The triangular numbers can also be shown using squares.

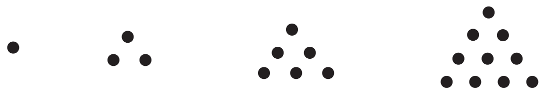


❶. Use interlocking cubes or squared paper to show the first seven triangular numbers.

Activity 1 – Teacher Edition

The Lesson Launch 10 minutes

Rather than beginning with a definition of a triangular number, we suggest that you launch this lesson by displaying a visual like this on the overhead projector.



Then pose these questions:

- 1, 3, 6, and 10 are triangular numbers; 2, 4, 5, 7, and 9 are not triangular numbers. Are there any triangular numbers between 11 and 20? If so, name one.
- What do you think is meant by the term, "triangular number"?

Encourage discussion until most students realize that a triangular number is a number that can be displayed in a triangular array of dots such that the first row has one dot and each succeeding row has one dot more than the previous row.

Initiating Activity 10 minutes

Have students work in pairs. Distribute interlocking cubes or scissors and a sheet of squared paper to each pair of students. Ask them to construct the first seven triangular numbers (as in Exercise ❶ on page 24). When they have successfully completed this exercise, ask them to answer the first question posed in the lesson launch above and invite them to justify their answers. Then distribute page 24 and have students, in turn, read to the class the story of the Pythagoreans. Ensure that all students understand what a triangular number is, and that the problem to be investigated in this unit is equivalent to: *What is the n^{th} triangular number?*

Small Group Activity 20 minutes

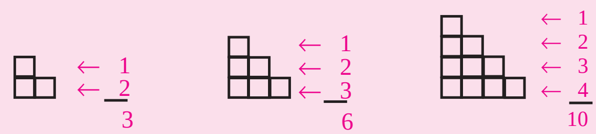
Have students work in the same pairs as in the initiating activity. Assign Exercises ❷ and ❸ on page 25. Students should discover in Exercise ❷ that we can obtain the fifth triangular number from the fourth by adding 5, and the sixth triangular number from the fifth by adding 6, and so on…. As students work on Exercise ❸, ensure that they understand that a *diagonal* is any line joining two (non-adjacent) vertices. It is vitally important that students describe *in words* the patterns they see in the numbers in the table.

Activity 1 – Student Page

How Many Dots in a Triangular Array with n Dots on each Side?

❷. a) What did the Pythagoreans mean by a "triangular number"?

b) The diagram shows how the second, third and fourth triangular numbers can be written as the sums 1 + 2, 1 + 2 + 3, and 1 + 2 + 3 + 4.

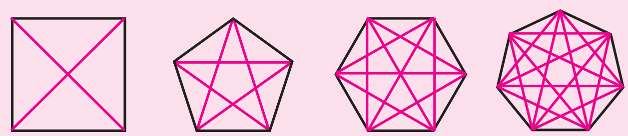


Describe how you can calculate the fifth triangular number if you know the fourth triangular number.

c) Fill in the missing numbers to show the first 10 triangular numbers.

1, 3, 6, 10, ☐, ☐, ☐, ☐, ☐,

❸. a) The following diagrams show regular polygons of 4, 5, 6, and 7 sides and all their diagonals. Record the number of sides and diagonals of each. Which polygons can you name?



number of sides ___ number of sides ___ number of sides ___ number of sides ___

number of diagonals _______ number of diagonals _______ number of diagonals _______ number of diagonals _______

b) Draw an octogon and all its diagonals. Record how many diagonals.

c) Using your answers to a) and b) make a table like this showing the number of diagonals of a polygon of n sides for n = 4, 5, 6, 7, and 8.

Number of Sides	Number of Diagonals	Triangular Number
4	2	3
5	5	6
6	9	10
•	•	•
•	•	•

d) Look for a pattern in your table. Conjecture how many diagonals in polygons of 9 and 10 sides. Draw diagrams to check your conjectures.

e) List 8 triangular numbers in your table, starting at 3. Conjecture a relationship between the numbers of diagonals and triangular numbers. Test your conjecture for polygons of 9 and 10 sides.

Closure

In this activity, students should discover how difficult it is to describe in words the relationship between the triangular numbers and the numbers of diagonals. For example, they might say that the number of diagonals is one less than the triangular number. However, you might ask, "one less than which diagonal number?" This leads them to the threshold of understanding the need for algebraic notation.

HOW MANY DOTS IN A TRIANGULAR ARRAY WITH n DOTS ON EACH SIDE?

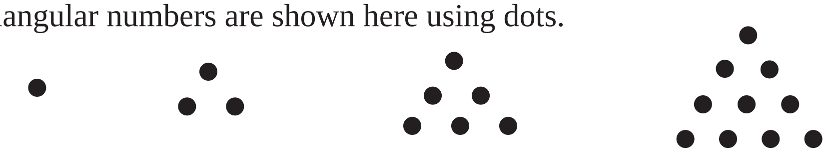
Our earliest records, from etchings in clay tablets to carvings in stone, reveal that humans through the ages have been fascinated with number patterns. The concept of number and the search for pattern has led to the development of the body of knowledge that we know today as *mathematics*. Contributions to mathematics have come from almost every culture and continent. In this unit, you will use number patterns to explore some number properties that have been investigated throughout history.

About 530 B.C. a group of philosophers and mathematicians in ancient Greece formed a secret society dedicated to the study of numbers and their properties. This secret society was known as the *Pythagoreans*, in honour of their distinguished leader, Pythagoras.

The Pythagoreans believed that the mysteries of the Universe could be explained in terms of number relationships. One of their members, Hippasus, proved that $\sqrt{2}$, the square root of 2, cannot be represented as a common fraction. When he shared his discovery with people outside the Pythagorean group, he was expelled for breaking his oath of secrecy. (Some accounts say he was killed by the Pythagoreans.)

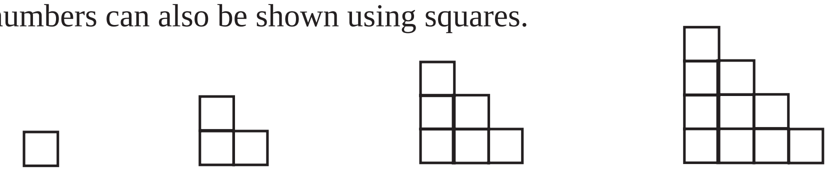
The Pythagoreans associated numbers with shapes. Numbers that can be represented by triangular arrays of dots (such that the n^{th} row contains *n* dots) were called *triangular numbers.* Numbers that can be displayed in *n* rows of *n* dots were called *square numbers*, and so on.

The first four triangular numbers are shown here using dots.



1 **3** **6** **10**

The triangular numbers can also be shown using squares.



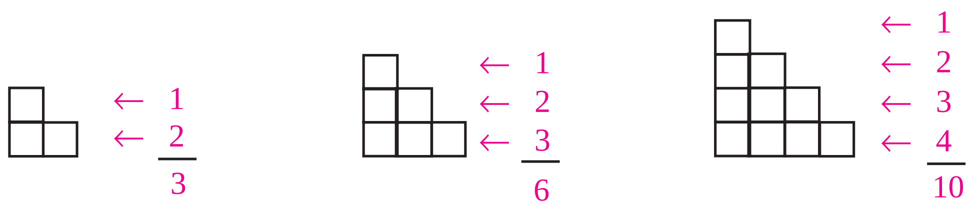
❶ Use interlocking cubes or squared paper to show the first seven triangular numbers.

The question that you will investigate in this unit is:

How Many Dots in a Triangular Array with n Dots on Each Side?

❷ a) What did the Pythagoreans mean by a "triangular number"?

b) The diagram shows how the second, third, and fourth triangular numbers can be written as the sums 1 + 2, 1 + 2 + 3, and 1 + 2 + 3 + 4.

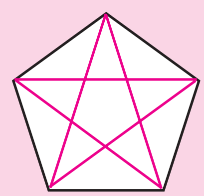


Describe how you can calculate the fifth triangular number if you know the fourth triangular number.

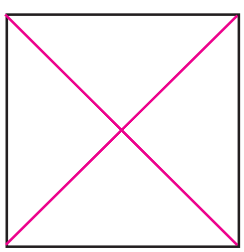
c) Fill in the missing numbers to show the first 10 triangular numbers.

1, 3, 6, 10, ☐, ☐ , ☐ , ☐ , ☐ , ☐ .

❸ a) The following diagrams show regular polygons of 4, 5, 6, and 7 sides and all their diagonals. Record the number of sides and diagonals of each. Which polygons can you name?

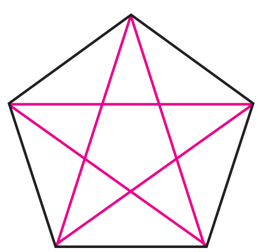

a pentagram

The Pythagoreans used the five-sided star, called a *pentagram*, as the symbol of their secret society. This is formed by drawing the diagonals of a pentagon.



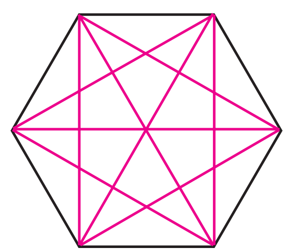
number of sides _______

number of diagonals ___



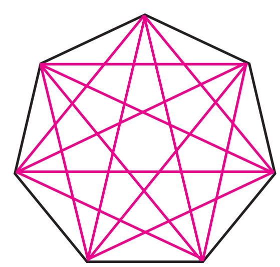
number of sides _______

number of diagonals ___



number of sides _______

number of diagonals ___



number of sides _______

number of diagonals ___

b) Draw an octogon and all its diagonals. Record how many diagonals.

c) Using your answers to a) and b), make a table like this showing the number of diagonals of a polygon of n sides for n = 4, 5, 6, 7, and 8.

d) Look for a pattern in your table. Conjecture how many diagonals in polygons of 9 and 10 sides. Draw diagrams to check your conjectures.

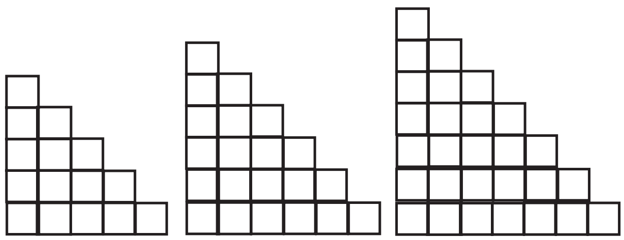
e) List 8 triangular numbers in your table, starting at 3. Conjecture a relationship between the numbers of diagonals and triangular numbers. Test your conjecture for polygons of 9 and 10 sides.

Number of Sides	**Number of Diagonals**	**Triangular Number**
4	2	3
5	5	6
6	9	10
•	•	•
•	•	•

Grade 7

Answer Key for Activity 1

❶ The first four triangular numbers are shown on page 24. The next three triangular numbers are:



15 21 28

❷ a) A triangular number is any number that can be displayed as a triangular array of objects such that the first row has one object and each row has one object more than the previous row.

b) The fourth triangular number is found by adding 4 to the third triangular number. The fifth triangular number is obtained by adding 5 to the fourth triangular number, so the fifth triangular number is 10 + 5 or 15.

c) The first 10 triangular numbers are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55

❸ The table gives the answers to ❸ a), b) & c).

Number of Sides	Number of Diagonals	Triangular Number
4	2	3
5	5	6
6	9	10
7	14	15
8	20	21
9	27	28
10	35	36

d) Students may discover a pattern in the middle column of the table. That is, the differences in successive terms of the sequence are 3, 4, 5, … From this pattern, they may conjecture that the number of diagonals in polygons of 9 and 10 sides are respectively 27 and 35. They can test this conjecture by drawing these polygons and counting the number of diagonals.

❸ e) The first 8 triangular numbers are:

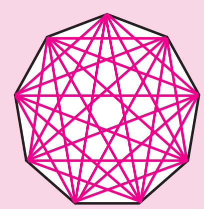
1, 3, 6, 10, 15, 21, 28, and 36

Students may discover that the numbers in the middle column are one less than the corresponding triangular numbers in the third column. By computing the 7th and 8th triangular numbers, 28 and 36, they may conjecture that the number of diagonals of polygons of 9 and 10 sides are respectively 27 and 35. They can verify this conjecture by comparing with their answer in ❸ d).

Teacher Note:

Some students will have difficulty counting the number of diagonals in polygons of 8 or more sides. To help them, ask whether the same number of diagonals terminate at each vertex. Once students realize that all vertices are the same, they need only count the number of diagonals that terminate in a single vertex. Then they multiply by the number of vertices to obtain the total number of "ends" of diagonals. Since each diagonal has two ends, they then divide by 2 to obtain the total number of diagonals.

For example, for the 9-sided polygon shown here, we observe that there are 6 diagonals terminating at each vertex. Therefore there is a total of 6×9 or 54 "ends" of diagonals. Since each diagonal has two ends, i.e., terminates at two vertices, it is counted twice, so we must divide 54 by 2 to obtain 27 diagonals.



Although it is *not* recommended that students be introduced to the algebraic formulation of this problem, the development is included here for your interest. In general, a polygon of n sides has $n - 3$ diagonals terminating at each vertex. The total number of diagonal "ends" is therefore $n(n - 3)$. However, since there are two diagonal "ends" for each diagonal, the actual number of diagonals is $n(n - 3)/2$. The number of diagonals of a polygon of $n + 2$ sides is therefore $(n + 2)(n - 1)/2$. Expansion of this expression reveals that it is one less than the n^{th} triangular number $n(n + 1)/2$.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING				
Identification and Extension of a Pattern. (exercises ❶ – ❸) P&A 7-1, 7-2, 7-3	•The pattern in the sequence of triangular numbers is extended incorrectly. •Conjectures are unreasonable.	•The pattern in the sequence of triangular numbers is not extended correctly to the specified number of terms. •Some reasonable conjectures are formulated, but there are minor errors.	•The pattern in the sequence of triangular numbers is extended correctly to the specified number of terms. •The conjectures are correct, but they are not checked or verified appropriately.	•In addition to Level 3, the student verifies that the conjectures are correct by drawing appropriate polygons and counting the diagonals in some organized way.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

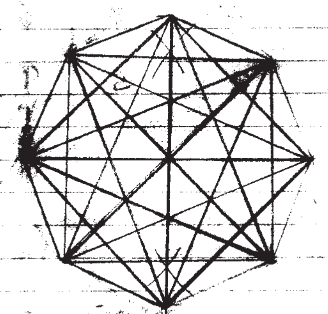
Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

Problem Solving: Identification & Extension of a Pattern

Level 2

③ b)



c)

Number of Sides	Number of diagnols	Triangular number
4	2	3
5	5	6
6	9	10
7	14	15
8	20	21

d) 9 / 10 | 26 / 34 | 27 / 35

E) 6, 10, 15, 21, 28, 36,

Keeps on adding 1 to number of sides as well as to number of diagnol

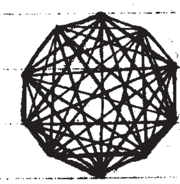
sides	4	5	6	7	8	9	10
diagnols	2	5	9	14	20	27	35

This student drew the octagon and its diagonals correctly. He also completed the table correctly up to $n = 8$. He miscalculated the triangular numbers 28 and 36 as 27 and 35. Since he understood that the number of diagonals was one less than the corresponding triangular number, this led him to conjecture the number of diagonals of a nonagon and decagon as 26 and 34 respectively. His conjecture was reasonable but contained a minor error (see the student's responses to ❸ d) and ❸ e)). Since he did not draw the polygons and count the diagonals, he did not discover his error. In Exercise ❸ e), the student did not extend the triangular numbers to the specified number of terms. He identified the relationship between the number of diagonals and the sequence of triangular numbers although he did not state the relationship explicitly.

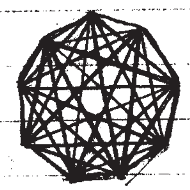
Problem Solving: Identification & Extension of a Pattern

Level 4

d) I think that in polygons of 9 sides there will be 27 diagonals and in polygons of 10 sides there will be 35 diagonals.



10 sides
35 diagonals



9 sides
27 diagonals

My conjecture was correct.

e)

Triangular Numbers	Number of Diagonals
3	2
6	5
10	9
15	14
21	20
28	27
36	35
45	44

The triangular numbers are always one number larger than the diagonal.

This student drew the nonagon and decagon and their diagonals correctly to verify her conjectures. She also completed the table correctly up to $n = 9$ for the triangular numbers. In ❸ e), she expressed correctly the relationship between the sequence of diagonal numbers and the sequence of triangular numbers. This student shows an understanding of the process of formulating a conjecture and then verifying it.

Activity 2 – Teacher Edition

An Ancient Pattern from the Chinese Culture

Expectations Addressed

P&A 7-1 describe patterns in a variety of sequences using the appropriate language and supporting materials.

P&A 7-2 extend a pattern, complete a table, and write words to explain the pattern.

P&A 7-3 recognize patterns and use them to make predictions.

P&A 7-6 present solutions to patterning problems and explain the thinking behind the solution process.

Context

In Activity 1, students were introduced to triangular numbers. Two closely related concepts that they need to review from that activity are:

- a triangular number is a positive integer that can be displayed as a triangular array of elements such that the first row has one element and each successive row has one more element than the previous row.
- the n^{th} triangular number is the sum of the integers from 1 to *n*.

Students may also have learned that the sequence of differences between successive triangular numbers is the sequence of natural numbers: 2, 3, 4, 5,…

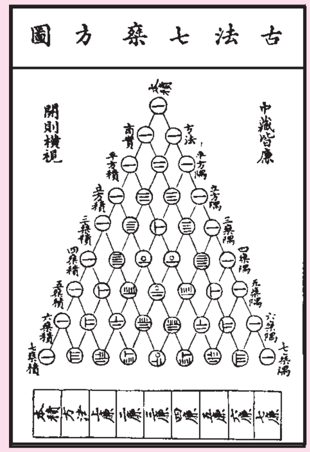
1, 3, 6, 10, 15, …

2, 3, 4, 5, …

In Exercise ❶ of Activity 2, students complete the template for Pascal's triangle using the property that any number in the triangle is the sum of the two numbers directly above it. In Exercises ❷ and ❸, they discover that the sequence formed by taking the second number in each row of Pascal's triangle is the sequence of positive integers. Furthermore, the sequence formed by taking the third number in each row (starting from row 3) is the sequence of triangular numbers. By tracing along the sequence of triangular numbers up to the n^{th} triangular number, students can find the sum of the integers from 1 to *n*. Finally, in Exercise ❹, students discover that the sum of any two triangular numbers is a perfect square, and the sum of the numbers in each row of Pascal's triangle is a power of 2.

Activity 2 – Student Page

An Ancient Pattern From the Chinese Culture

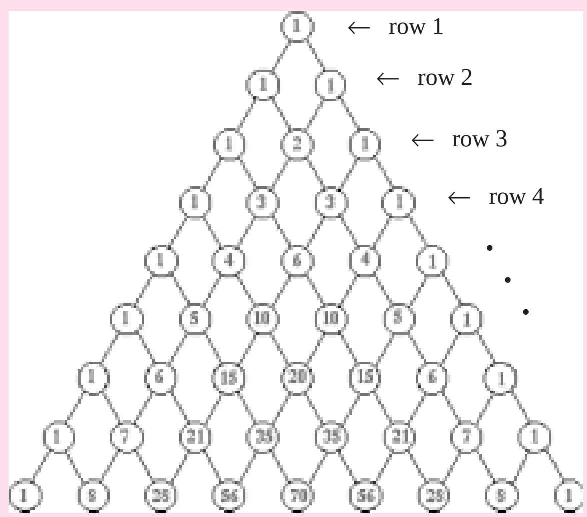


One of the most important contributions from the Chinese civilization was a book *Ssu-yüan yü-chien* (Precious Mirror of the Four Elements) written by mathematician Chu Shih-chieh in 1303. At the front of that book appears the triangular array of symbols shown in the illustration.

When these Chinese symbols are replaced with Hindu-arabic numerals, this triangular array becomes the array shown below, known as *Pascal's triangle*. It was named in honour of the French Mathematician Blaise Pascal (1623-1662) who re-discovered it when he was thirteen years old. We see here only the first nine rows.

Compare the two arrays. Can you identify what symbols the Chinese used to represent each of the numerals from 1 to 10? Write the Chinese symbol beside the corresponding Hindu-arabic numeral.

Describe any number patterns you can see in Pascal's triangle. Check along horizontal, vertical, and diagonal lines.



Activity 2 – Teacher Edition

The Lesson Launch 5 minutes

Before launching the lesson, ensure that each student has a copy of pages 32 and 33 and of *A Template for Pascal's Triangle* (page 56). Have students follow along as you read the historical note on page 32. Ask them to compare the Chinese and modern versions of Pascal's triangle. Then launch the lesson with questions such as the following:

- What is the Chinese numeral for 6? for 10? for 15?
- How did the Chinese make the numeral for 15 out of the numerals for 10 and 5?
- How can each number in Pascal's triangle be calculated from the numbers in the row above it?

Once students understand the answer to the last of these questions, they should complete individually *A Template for Pascal's Triangle* (see Exercise ❶ b). This will ensure that each one understands the structure of Pascal's triangle before attempting to investigate its properties.

Individual Activity 40 minutes

Once it is clear that students have successfully completed Pascal's triangle (only the bottom row needs to be checked), ask them to work on Exercises ❷ through ❹ individually. Some students will have difficulty with terminology such as "numbers from 1 to *n*." Sometimes it helps to suggest that *n* is an abbreviation for "anything." Emphasize that *n* is a variable and that a variable is a symbol that can be replaced by any one of a given set of numbers. Ask such students, *What is the sum of the numbers from* 1 *to n when n takes the value* 4? 7? 9?

When students have completed Exercises ❷ through ❹, ask them to explain how to find the n^{th} triangular number in Pascal's triangle. Encourage them to use terminology such as, *The n^{th} triangular number is the third number in the n^{th} row of Pascal's triangle.*

Caution students that Exercise ❺ is a challenging problem, usually given to students at a higher grade level, but encourage them to try it themselves and get help if necessary from friends or parents. If they work backwards, filling in first the circles closest to home, they may discover that the process for constructing these numbers is the same as for constructing Pascal's triangle, and if they rotate the grid by 45˚ they will indeed obtain the first nine rows of Pascal's triangle with the number 70 in the START position.

Activity 2 – Student Page

An Ancient Pattern From the Chinese Culture

❶ a) Choose any number in Pascal's triangle that has two numbers directly above it. Compare the sum of those two numbers with the number you chose. What do you discover? Does this depend on the number you chose?

b) Use what you discovered in Part a) to fill in the missing numbers in the template for Pascal's triangle. (Your teacher will provide this.)

c) Describe any symmetry you see in Pascal's triangle.

❷ a) List in order the second number in each row of Pascal's triangle (starting with the second row).

1, 2, ☐, ☐, ☐, ☐, …

↑ 2nd number in row 2 ↑ 2nd number in row 3 ↑ 2nd number in row 4

Describe any pattern in this list.

b) Write the sum of the first six numbers in your list. Locate this number in Pascal's triangle and circle it.

c) Write the sum of the first seven numbers in your list. Locate this number in Pascal's triangle and circle it.

d) Describe how to calculate the sum 1 + 2 + 3 + 4 + 5 + 6 + 7 by using Pascal's triangle. Use your completed template from Exercise ❶ to find the sum of the whole numbers from 1 to 10.

❸ a) List the third number in each row of Pascal's triangle (starting with the third row). What name did the Pythagoreans give to these numbers?

b) Locate the line of triangular numbers in Pascal's triangle. In what rows are the 5^{th}, the 6^{th} and the n^{th} triangular numbers?

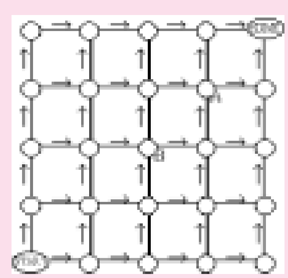
c) Explain how the sum of the numbers from 1 to *n* is related to the n^{th} triangular number.

❹ a) Choose two consecutive numbers in your list of triangular numbers. Add them. What special characteristic has this number?

b) Make a list showing the sum of the numbers in each row of Pascal's triangle. In what way are all these sums alike?

❺ **Taxicab Geometry**

a) Record the number of routes from point A to HOME if you must always travel either north or east on each road. (A road is any line segment joining two circles.) Write this number in the circle beside A.



b) Write in each circle the number of routes from that circle to HOME. How many paths from B to HOME?

c) Describe any pattern you discover. Use your pattern to determine how many routes from START to HOME.

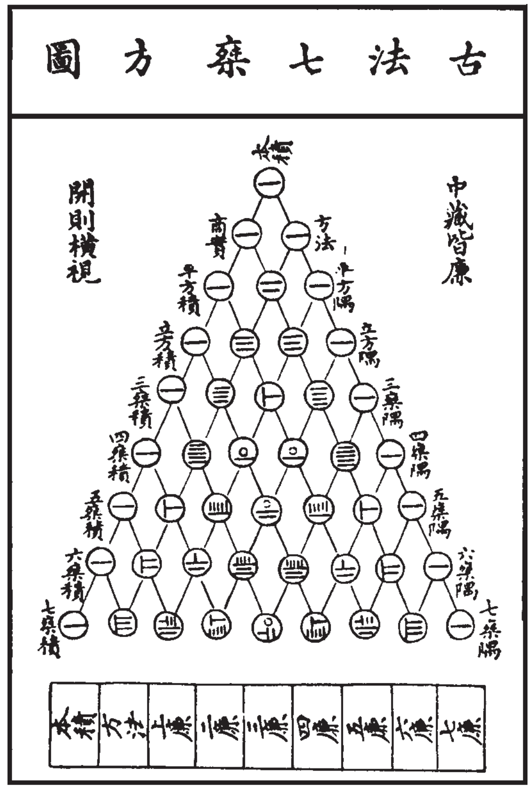
Closure

To consolidate and extend the students' facility in thinking algebraically, explain that the last five minutes of class will be an "Alge-speak session." They must respond in algebraic terms. Then present questions such as:

- *What number comes after n? before n?*
- *What row comes after the n^{th} row? two rows before the n^{th} row?*

AN ANCIENT PATTERN FROM THE CHINESE CULTURE

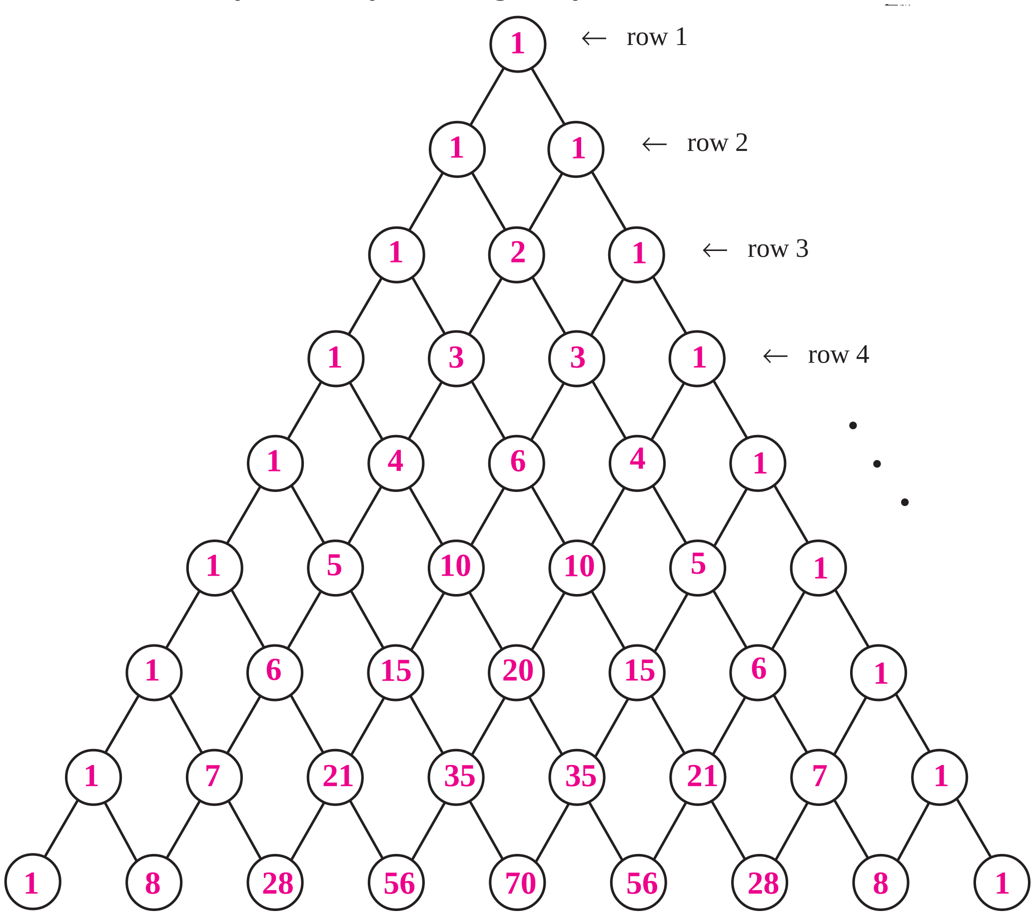
One of the most important contributions from the Chinese civilization was a book *Ssu-yüan yü-chien* (Precious Mirror of the Four Elements) written by mathematician Chu Shih-chieh in 1303. At the front of that book appears the triangular array of symbols shown in the illustration.



When these Chinese symbols are replaced with Hindu-arabic numerals, this triangular array becomes the array shown below, known as *Pascal's triangle*. It was named in honour of the French mathematician Blaise Pascal (1623–1662) who re-discovered it when he was thirteen years old. We see here only the first nine rows.

Compare the two arrays. Can you identify the symbols the Chinese used to represent each of the numerals from 1 to 10? Write the Chinese symbol beside the corresponding Hindu-arabic numeral.

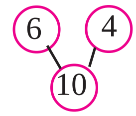
Describe any number patterns you can find in Pascal's triangle. Check horizontally, vertically, and diagonally.



AN ANCIENT PATTERN FROM THE CHINESE CULTURE

❶ a) Choose any number in Pascal's triangle that has two numbers directly above it. Compare the sum of those two numbers with the number you chose. What do you discover? Does this depend on the number you chose?
b) Use what you discovered in Part a) to fill in the missing numbers in the template for Pascal's triangle. (Your teacher will provide this.)
c) Describe any symmetry you see in Pascal's triangle.

Example



← 6 and 4 are directly above 10.

❷ a) List in order the second number in each row of Pascal's triangle (starting with the second row).

1, 2, ☐, ☐, ☐, ☐, ☐, ☐ …

↑ 2nd number in row 2 ↑ 2nd number in row 3 ↑ 2nd number in row 4

Describe any pattern in this list.

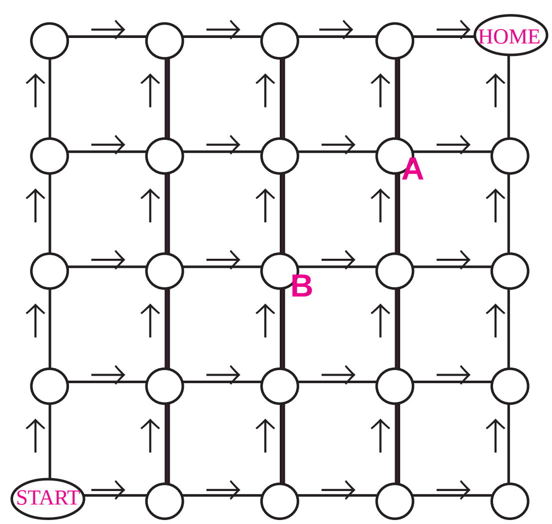
b) Write the sum of the first six numbers in your list. Locate this number in Pascal's triangle and circle it.
c) Write the sum of the first seven numbers in your list. Locate this number in Pascal's triangle and circle it.
d) Describe how to calculate the sum 1 + 2 + 3 + 4 + 5 + 6 + 7 by using Pascal's triangle. Use your completed triangle from Exercise ❶ to find the sum of the whole numbers from 1 to 10.

❸ a) List the third number in each row of Pascal's triangle (starting with the third row). What name did the Pythagoreans give to these numbers?
b) Locate the line of triangular numbers in Pascal's triangle. In what rows are the 5^{th}, the 6^{th} and the n^{th} triangular numbers?
c) Explain how the sum of the numbers from 1 to n is related to the n^{th} triangular number.

❹ a) Choose two consecutive numbers in your list of triangular numbers. Add them. What special characteristic does this number have?
b) Make a list showing the sum of the numbers in each row of Pascal's triangle. In what way are all these sums alike?

❺ TAXICAB GEOMETRY

a) Record the number of routes from point A to HOME if you must always travel either north or east on each road. (A road is any line segment joining two circles.) Write this number in the circle beside A.
b) Write in each circle the number of routes from that circle to HOME. How many paths from B to HOME?
c) Describe any pattern you discover. Use your pattern to determine how many routes from START to HOME.



Grade 7
Answer Key for Activity 2

❶ a) Every number in Pascal's triangle that has two numbers directly above it is the sum of those two numbers.

b) The template is found on page 56. The ninth, tenth and eleventh rows are respectively:

1, 9, 36, 84, 126, 126, 84, 36, 9, 1
1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1
1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1

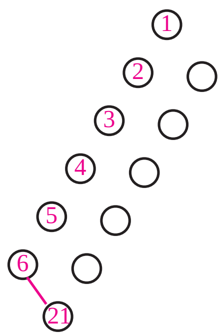
c) Pascal's triangle is symmetric about the vertical line through its top vertex and running through the numbers 1, 2, 6, 20, …

❷ a) The sequence is:

1, 2, 3, 4, 5, 6, 7, 8,…

This is the sequence of positive integers.

b) The sum of the first 6 positive integers is:
1 + 2 + 3 + 4 + 5 + 6 = 21
which is located in the row following the sequence
1, 2, 3, …, 6 and one column to the right.



c) The sum of the first 7 integers is 1 + 2 + … + 7 or 28. We locate this in the 8th row and 3rd column of Pascal's triangle and we circle it.

d) As in Part b), we can find the sum of the first seven integers by locating the number 28 in the row below the sequence 1, 2, 3, 4, 5, 6, 7 and one column right of 7.

We find the sum of the first ten positive integers by scanning Pascal's triangle for the number in the 11th row that is one column to the right of 10. It is the number 55. Therefore the sum of the integers from 1 to 10 is 55.

e) The sum of the integers from 1 to n is the nth triangular number. This can be verified directly using the definition of the nth triangular number.

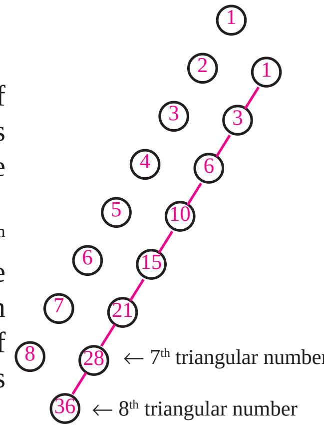
Note: In the template on page 56, we ask students to check the truth of the following famous theorem in Number Theory.

Every integer is the sum of one, two, or three triangular numbers.

A formal proof is beyond students at this level, but they can have fun challenging each other to find the triangular numbers that sum to a given integer.

❸ a) The third number of each row of Pascal's triangle is the sequence of triangular numbers.

b) The fifth, sixth and nth triangular numbers are in the seventh, eighth and $(n + 2)$th rows of Pascal's triangle, as shown in the diagram.



❹ a) The sums of pairs of consecutive triangular numbers is shown here.

1 3 6 10 15 21 28 36

4 9 16 25 36 49 64

We observe that these sums of consecutive triangular numbers are (perfect) squares.

b) The sums of the numbers in the first 8 rows of Pascal's triangle are respectively:

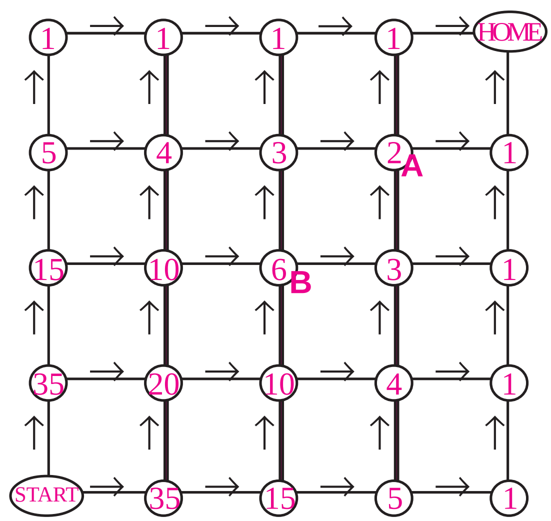
1, 2, 4, 8, 16, 32, 64, 128

Written as powers of 2, these sums are:

2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7.

In general, the sum of the numbers in the nth row of Pascal's triangle is 2^{n-1}.

❺ a) When we record the number of routes from each point to HOME, we obtain this diagram.



b) There are 6 routes from B to HOME.

c) Rotating this diagram 45˚ yields Pascal's triangle. The number of routes from START to HOME is 35 + 35 = 70.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1- 8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Application of a Pattern to the Solution of a Problem** (exercises ❷ d), e), ❺) P&A 7-2, 7-6	•Significant difficulty identifying patterns.	•Patterns are usually identified correctly, but not correctly applied to the solution of a problem.	•Patterns are always identified correctly, but not always applied correctly to the solution of a problem.	•Patterns are always identified correctly and applied correctly to the solution of a problem.
COMMUNICATION **Description and Explanation of a Pattern** (exercises ❶, ❷, ❹ & ❺) P&A 7-1, 7-2, 7-3	•Statements are unclear and contain major errors and/or omissions.	•Statements are clear but incomplete and/or have several minor errors or omissions.	•Statements are clear and have a few errors or omissions.	•Statements are clear and concise with little or no errors or omissions.

WHAT YOU MIGHT SEE

Problem Solving: Application of a Pattern to the Solution of a Problem

Level 1

5) a) There are two paths from point A to home. There are two paths from Point B to home.

b) There are 2 paths from B to home.

c) There are 2 paths from Start to home.

This student had difficulty identifying most of the patterns in this activity. In the response to Exercise ❺, it is clear that the student has not understood what is meant by a *route* and how the numbers at the grid points are related to Pascal's triangle.

Level 3

5a) From A to Home, there is only two paths.

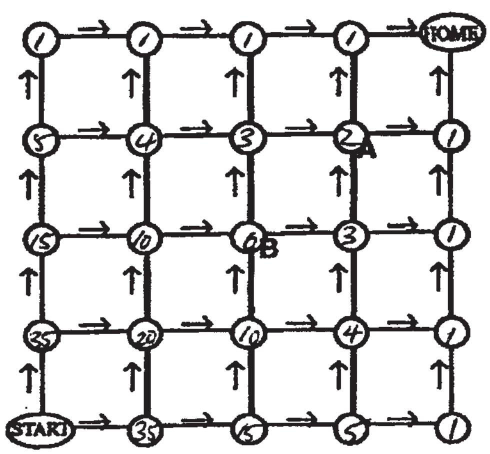
b) From B to Home, there are 6 paths to it.

c) The diagram shows the same pattern as Pascal's triangle.

This student identified correctly all the patterns in this activity. In Exercise ❺, the student calculated correctly enough of the numbers at the grid points to identify the pattern with Pascal's triangle. However, this pattern was not used to calculate the number of paths from START to HOME.

Level 4

5 a)



b)

c) It is actually Pascal's triangle except the corners of the triangle are cut off. 70 paths

This student identified correctly all the patterns in this activity. In Exercise ❺, the student calculated correctly all the numbers at the grid points. The identification of the pattern with Pascal's triangle was correctly applied to determine the number of paths in ❺ c). Although explicit responses were not given in ❺ a) and b), it is clear from the completed diagram and from the response in ❺ c) that the student knew the answers.

WHAT YOU MIGHT SEE

Communication: Description and Explanation of a Pattern

Level 2

4a) 3 and 6. 3+6=9. It is a common factor of 3. No, it is not true because if you take 7 and add it to 9 your answer is 16. This number has nothing in common with the numbers 9 and 7.

b)

row = 1	= 1
row = 2	= 2
" " = 3	= 4
" " = 4	= 8
" " = 5	= 16
" " = 6	= 32
" " = 7	= 64
" " = 8	= 128
" " = 9	= 256

Every time you have to add the number you have with itself to get ~~the next~~ number (Eg. 16 + 16=32) 32 +32 = 64

In Exercise ❹, the student has not discovered that the sum of two consecutive triangular numbers is a square. She has looked only at the two triangular numbers 3 and 6 and concluded that the sums of consecutive triangular numbers is a multiple of 3. However, in her statement, "It is a common factor of 3," she has used the terminology incorrectly . This is a minor error. Her observation in 4b) is that each number is double the previous number. She has stated this clearly and provided an example for clarification.

Level 4

2 d) By moving down the Third diagonal row of Pascal's triangle, until you reach the last number of you addition sentence. Move downwards towards the center one space and you will reach your sum.

4 a) **Two consecutive numbers: 6, 10.**
6+10=16 The special characteristic of this number is that it is a square number. Yes, it is true no matter what pair of consecutive numbers you choose.

b) **1, 2, 4, 8, 16, 32, 64, 128, 356.**
The sum of the row is double the sum of the previous row.

In Exercise ❷ d), the student has explained clearly and correctly how to find the sum of the numbers from 1 to *n* in Pascal's triangle. In Exercise ❹, the student has discovered that the sum of two consecutive triangular numbers is a square. He has further indicated that this property is true for all pairs of consecutive triangular numbers. The response in ❹ b) is also correct and clear. This high level of articulation was evident throughout this student's work on Activity 2.

Activity 3 – Teacher Edition

From Patterns to Algebra

Expectations Addressed

P&A 7-1 describe patterns in a variety of sequences using the appropriate language and supporting materials.

P&A 7-2 extend a pattern, complete a table, and write words to explain the pattern.

P&A 7-3 recognize patterns and use them to make predictions.

P&A 7-4 interpret a variable as a symbol that may be replaced by a given set of numbers.

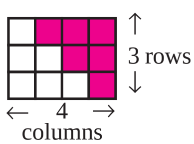
P&A 7-5 write statements to interpret simple formulas.

P&A 7-6 present solutions to patterning problems and explain the thinking behind the solution process.

Context

This activity involves students in a hands-on activity with squared paper that helps them develop an algebraic expression for the n^{th} triangular number. This development hinges on the discovery that a rectangular array of n rows and $n + 1$ columns can be divided into two identical (i.e. congruent) triangular arrays with n squares on each side.

Example with $n = 3$



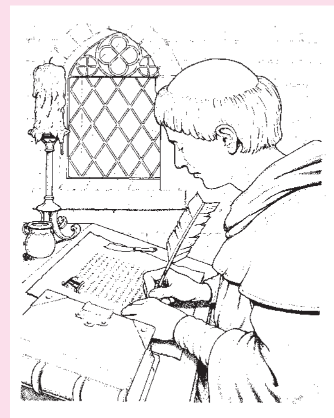
By cutting out of squared paper (see p. 93) several rectangular arrays of dimension n by $n + 1$ (for various values of n) and generalizing from these specific values of n, students will discover that a rectangular array of dimension n by $n + 1$ is composed of:

- $n(n + 1)$ squares.
- two congruent triangular arrays with n squares per side.

From these two facts, students can deduce that the number of squares in a triangular array with n squares on each side is half of $n(n + 1)$, i.e., $n(n + 1)/2$. In other words, the n^{th} triangular number is given by the algebraic expression $n(n + 1)/2$. In Exercise ❹, the student is reminded that the n^{th} triangular number is the sum of the numbers from 1 to n and so it follows that $1 + 2 + 3 + \ldots + n = n(n + 1)/2$. A knowledge of this formula enables students to compute the n^{th} triangular number or the sum of the numbers from 1 to n, for any value of n. For many students, this may be their first glimpse into the power of algebra!

Activity 3 – Student Page

From Patterns to Algebra

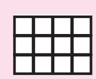


After the collapse of the Roman Empire in 476, Europe entered a period of history called the *Dark Ages*. The ancient civilizations that had encouraged learning had been replaced by nomadic people. The mathematics of the ancient Greeks would have been lost forever were it not for the monks who copied the Greek manuscripts to preserve them for future generations. Mathematical activity during the Dark Ages was maintained outside Europe by the Arabs, the Hindus and the Chinese. The Hindus invented the concept of the number zero and developed, between 800 A.D. and 1200 A.D., the Hindu-Arabic numeration system that we use today. Around 800 A.D., Arab mathematician Mohammed ibn-Musa al-Khowarizmi wrote a book on arithmetic operations and equations titled *Al-jabr wa'l muqabalah*. From this title came the word "algebra" that we use to describe the symbolic form of arithmetic. Our word "algorithm," used to describe an arithmetic procedure, derives from his name, "al-Khowarizmi." Near the end of the eleventh century, Persian poet and mathematician Omar Khayyam claimed to have discovered a pattern for expanding powers – a pattern that some historians now believe was Pascal's triangle!

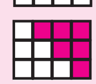
In the 1500s, mathematical exploration reappeared in Europe and modern algebraic notation evolved. Pascal's triangle (presented by Chinese mathematician Chu Shih-chieh in 1303) reappeared in 1653, in a publication of French mathematician Blaise Pascal. Using this triangle, Pascal proved some important mathematical relationships involving sums of powers of whole numbers. In this activity, you will discover one of these relationships as well as an algebraic formula for the n^{th} triangular number.

❶. Use squared paper.

a) Cut out a 3×4 rectangular array as shown.



b) Colour the squares to divide your rectangle into two identical triangular arrays as shown.



c) Record the number of squares in each triangular array and the total number of squares in the rectangular array.

d) Describe how the number of squares in a 3×4 rectangular array is related to the number of squares in a triangular array with 3 squares on each side.

Activity 3 – Teacher Edition

The Lesson Launch

10 minutes

Launch this lesson with another "Alge-speak session" (see Closure to Activity 2, page 31) by asking questions such as:

- *What are the dimensions of a rectangular array of squares with n rows and n + 1 columns?*
- *How many squares are there in such an array?*
- *How many squares would be in such an array if it were divided into two congruent pieces?*

Before proceeding with the lesson, ensure that the students are able to answer these questions. Those students who are having difficulty understanding the concept of a variable and expressing the general case in terms of n should be paired with another student for peer tutoring. Allow a few minutes for this. To those students who continue to have difficulty, distribute copies of *Pattern Sleuthing – Warm Up* (see pages 16 and 17 for template). Have them complete this during class. While the other students are completing the next activity, check to ensure that the students involved in Pattern Sleuthing are able to express the n^{th} terms of the sequences algebraically.

Individual Activity

25 minutes

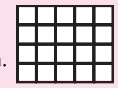
To each of the students who are *not* working on the *Pattern Sleuthing–Warm Up*, hand out student Activity 3 *From Patterns to Algebra* (pp. 40–41). Have students read the historical section on page 40 and ensure that they understand the terms used. Distribute scissors and a sheet of squared paper to each of these students. Ask the students to complete exercises ❶ through ❹ and challenge them to see who can find an algebraic expression for the n^{th} triangular number. Circulate around the class, checking particularly on each student's ability to answer correctly Exercises ❸ c) and ❹. Check also on the students working on the *Pattern Sleuthing*.

When you discover the first student to achieve the expression $n(n + 1)/2$ for the n^{th} triangular number, praise the student's triumph and further challenge the student to sum the integers from 1 to 100. Encourage the remaining students to continue to search for the expression. When most students have discovered the expression or appear to be reaching frustration, terminate the activity. Invite the student who first discovered the formula to explain to the class on the blackboard or overhead projector how he or she discovered it.

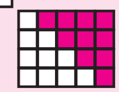
Activity 3 – Student Page

From Patterns to Algebra

❷. Use squared paper.



a) Cut out a 4×5 rectangular array as shown.



b) Colour the squares to divide your rectangle into two identical triangular arrays as shown.

c) Record the number of squares in each triangular array and the total number of squares in the rectangular array.

d) Describe how the number of squares in a 4×5 rectangular array is related to the number of squares in a triangular array with 4 squares on each side.

e) Describe the relationship between the number of squares in a triangular array with 5 squares on each side and the number of squares in a 5×6 rectangular array.

❸. Use Exercise ❷ to help you fill in the blanks in each statement.
The number of squares in a triangular array:

a) with 4 squares on each side is: $\frac{\square \times \square}{2}$.

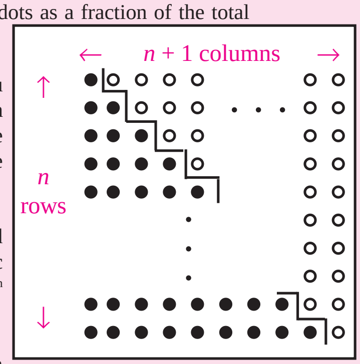
b) with 5 squares on each side is: $\frac{\square \times \square}{2}$.

c) with 6 squares on each side is: $\frac{\square \times \square}{2}$.

Make a conjecture about the number of squares in a triangular array with n squares on each side.
Explain why you believe your conjecture is true.

❹. a) The diagram shows a rectangular array of n rows of dots with $n + 1$ dots in each row. Write the number of dots in the array in terms of n.

b) Write the number of black dots as a fraction of the total number of dots in the array.



c) Use the relationship you found in ❹ b) to write an algebraic expression for the number of black dots in the rectangular array.

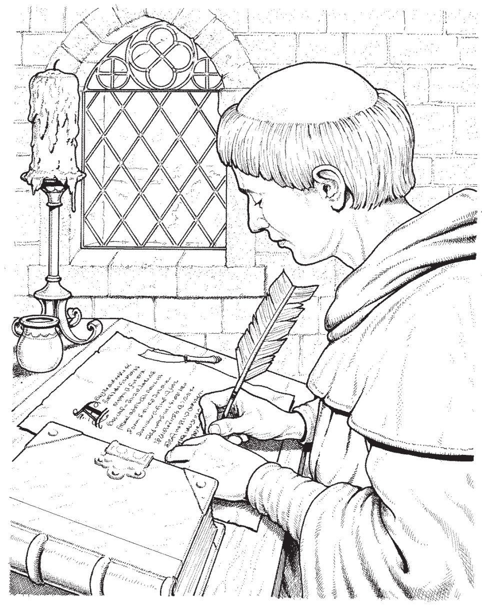
d) Use the expression you found in Part c) to write an algebraic expression for the n^{th} triangular number.

e) Write an algebraic expression for the sum of the integers from 1 to n, i.e. $1 + 2 + 3 + \ldots + n$.

Closure

By the end of this lesson, it is important that all or almost all students know how to substitute various values of n into the formula to compute the n^{th} triangular number. To reinforce this learning, invite various individuals to compute triangular numbers at the blackboard for various values of n. For homework, challenge them to find the sum of the numbers from 1 to n when $n = 100$ and $n = 1000$. Those who complete the first challenge by merely adding with a calculator will discover in the second challenge that algebra is indeed a powerful tool.

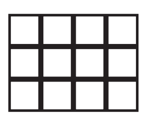
From Patterns to Algebra



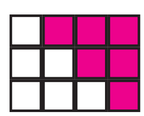
After the collapse of the Roman Empire in 476, Europe entered a period of history called the *Dark Ages*. The ancient civilizations that had encouraged learning were replaced by nomadic people. The mathematics of the ancient Greeks would have been lost forever if monks hadn't copied the Greek manuscripts to preserve them for future generations. Mathematical activity during the Dark Ages was maintained outside Europe by the Arabs, the Hindus, and the Chinese. The Hindus invented the concept of the number zero and developed, between 800 A.D. and 1200 A.D., the Hindu-Arabic numeration system that we use today. Around 800 A.D., Arab mathematician Mohammed ibn-Musa al-Khowarizmi wrote a book on arithmetic operations and equations titled *Al-jabr wa'l muqabalah*. From this title came the word "algebra" that we use to describe the symbolic form of arithmetic. Our word "algorithm," used to describe an arithmetic procedure, derives from his name, "al-Khowarizmi." Near the end of the eleventh century, Persian poet and mathematician Omar Khayyam claimed to have discovered a pattern for expanding powers – a pattern that some historians now believe was Pascal's triangle!

In the 1500s, mathematical exploration reappeared in Europe and modern algebraic notation evolved. Pascal's triangle (presented by Chinese mathematician Chu Shih-chieh in 1303) reappeared in 1653, in a publication of French mathematician Blaise Pascal. Using this triangle, Pascal proved some important mathematical relationships involving sums of powers of whole numbers. In this activity, you will discover one of these relationships as well as an algebraic formula for the n^{th} triangular number.

❶ Use squared paper.



a) Cut out a 3×4 rectangular array as shown.



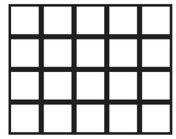
b) Colour some of the squares to divide your rectangle into two identical triangular arrays as shown.

c) Record the number of squares in each triangular array and the total number of squares in the rectangular array.

d) Describe how the number of squares in a 3×4 rectangular array is related to the number of squares in a triangular array with 3 squares on each side.

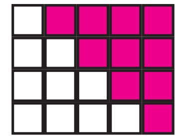
FROM PATTERNS TO ALGEBRA

❷ Use squared paper.



a) Cut out a 4×5 rectangular array as shown.

b) Colour the squares to divide your rectangle into two identical triangular arrays as shown.



c) Record the number of squares in each triangular array and the total number of squares in the rectangular array.

d) Describe how the number of squares in a 4×5 rectangular array is related to the number of squares in a triangular array with 4 squares on each side.

e) Describe the relationship between the number of squares in a triangular array with 5 squares on each side and the number of squares in a 5×6 rectangular array.

❸ Use Exercise ❷ to help you fill in the blanks in each statement.

The number of squares in a triangular array

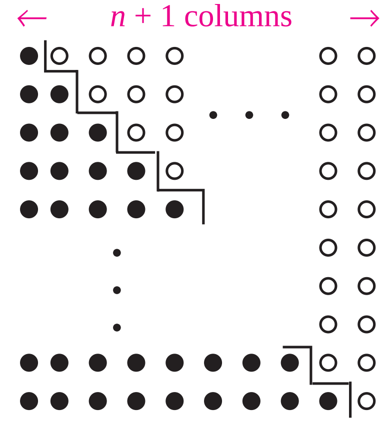
a) with 4 squares on each side is $\dfrac{\square \times \square}{2}$.

b) with 5 squares on each side is $\dfrac{\square \times \square}{2}$.

c) with 6 squares on each side is $\dfrac{\square \times \square}{2}$.

Make a conjecture about the number of squares in a triangular array with n squares on each side.
Explain why you believe your conjecture is true.

❹ a) The diagram shows a rectangular array of n rows of dots with $n + 1$ dots in each row. Write the number of dots in the array in terms of n.



b) Write the number of black dots as a fraction of the total number of dots in the array.

c) Use the relationship you found in ❹ b) to write an algebraic expression for the number of black dots in the rectangular array.

d) Use the expression you found in ❹ c) to write an algebraic expression for the n^{th} triangular number.

e) Write an algebraic expression for the sum of the integers from 1 to n, i.e., $1 + 2 + 3 + \ldots + n$.

Grade 7

Answer Key for Activity 3

❶ c) Each triangular array contains 6 squares and the rectangular array contains 12 squares.

d) The number of squares in a 3×4 rectangular array is double the number of squares in a triangular array with 3 squares on each side.

❷ c) Each triangular array contains 10 squares and the 4×5 rectangular array contains 20 squares.

d) The number of squares in a 4×5 rectangular array is double the number of squares in a triangular array with 4 squares on each side.

❸ From Exercise ❷:

The number of squares in a triangular array

a) with 4 squares on each side is $\frac{\boxed{4} \times \boxed{5}}{2}$.

b) with 5 squares on each side is $\frac{\boxed{5} \times \boxed{6}}{2}$.

c) with 6 squares on each side is $\frac{\boxed{6} \times \boxed{7}}{2}$.

Students are expected to conjecture the algebraic expression

$$\frac{\boxed{n} \times \boxed{n+1}}{2}$$

for the number of squares in a right triangle with n squares on each side, by extending the pattern above. This is an example of inductive reasoning. It is important that students be involved in such experiences before they embark upon the kind of deductive reasoning required in Exercise ❹. In their explanation of their conjectures, some students might jump directly to the deduction by observing that a right triangle with n squares on each side is half an n by $n + 1$ rectangle. By observing that such a triangle has one square in the smallest row, two squares in the next row, etc., the student can deduce that the expression above is also the n^{th} triangular number.

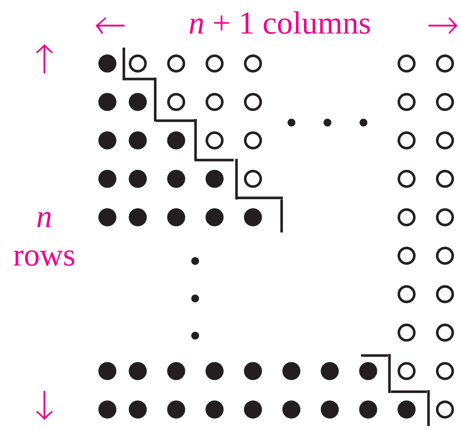
❹ a) The rectangular array has $n \times (n + 1)$ dots.

b) The number of black dots is half the total number of dots.

c) From Parts a) and b), the number of black dots is $n \times (n + 1)/2$.

d) The n^{th} triangular number is given by: $n \times (n + 1)/2$ or $n(n + 1)/2$.

e) Since the n^{th} triangular number is the same as the sum of the integers from 1 to n, then: $1 + 2 + 3 + \ldots + n = n(n + 1)/2$.



Teacher Note

You will note that Activity 3 is more structured and less open-ended than most activities. In the development of the formula for the n^{th} triangular number, it is necessary to provide a significant amount of guidance to ensure that most students understand the logical steps that lead to the formula. It is not important whether the students know the formula or whether they are able to derive it. The goal here is to help them understand how they can generalize their reasoning in several specific cases to apply to the general case, and hence all cases.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 3

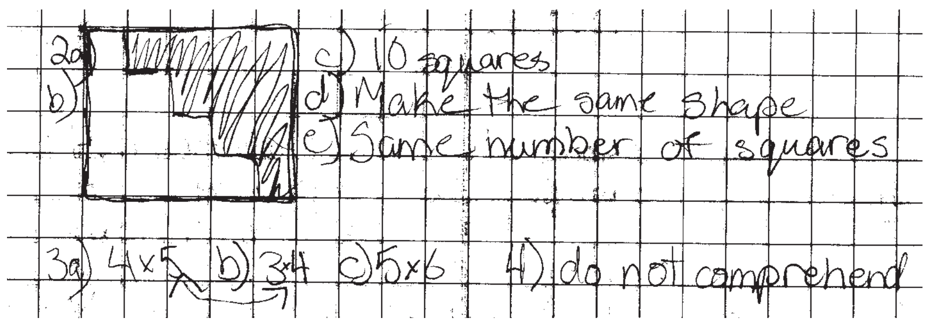
	Level 1	Level 2	Level 3	Level 4
CONCEPTS **Understanding of a Variable as a Generalized Number** (exercises ❸ & ❹) P&A 7-4, 7-7	•There is little or no evidence of the ability to use a variable to generalize a pattern.	•Some evidence of the ability to use a variable to generalize a pattern.	•A variable is usually used appropriately to generalize a pattern.	•A variable is almost always used appropriately to generalize a pattern.
APPLICATION **Translation of Statements into Algebraic Expressions** (exercises ❸ & ❹) P&A 7-8	•Gives incorrect or no algebraic expression in most instances.	•Gives incorrect or no algebraic expression in several instances.	•Gives correct algebraic expression in most instances.	•Gives correct algebraic expression in almost all instances.

Note: In making the transition from arithmetic to algebra, the student must be able to interpret a variable as a symbol that stands for an arbitrary positive integer (see expectation P&A 7-4, p. 20). This understanding is prerequisite to the correct translation of simple statements into algebraic expressions (see P&A 7-8, p. 21). For this reason, you will find that the scoring guide above will usually assign to a student the same level for both the concept and the application. However, as the statements to be translated into algebra become more complex (e.g., in Activity 4), students who understand the concept of a variable may not necessarily have success in formulating the required algebraic statements.

WHAT YOU MIGHT SEE

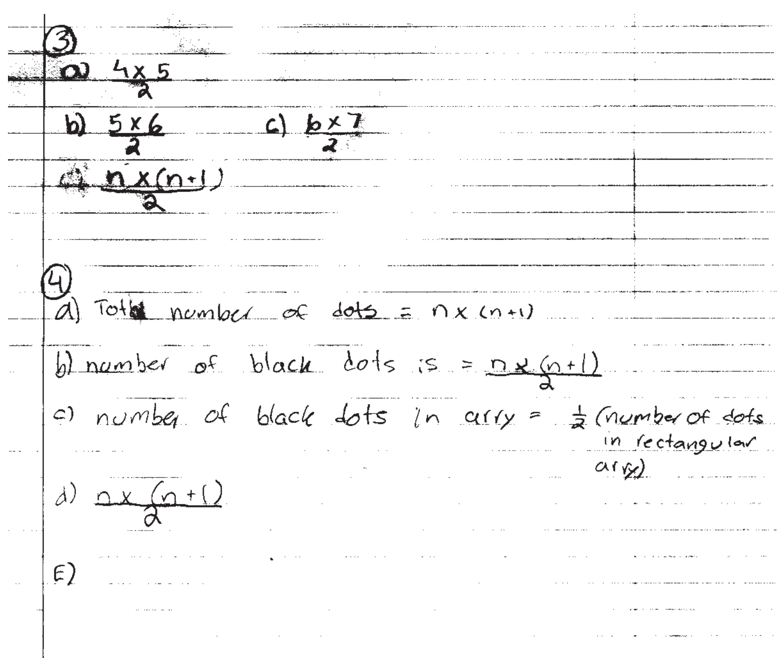
Understanding of Concepts: Understanding of a Variable as a Generalized Number

Level 1



In Exercise ❸ a), b), and c), the student has not divided by 2 to obtain the correct number of squares in the triangular arrays. When it comes to generalizing the number of squares in a triangular array with *n* squares on each side (Exercise ❸ d) and ❹), the student responds, “Do not comprehend.” There is no evidence in any of the student’s work of an ability to use a variable to generalize a pattern.

Level 3

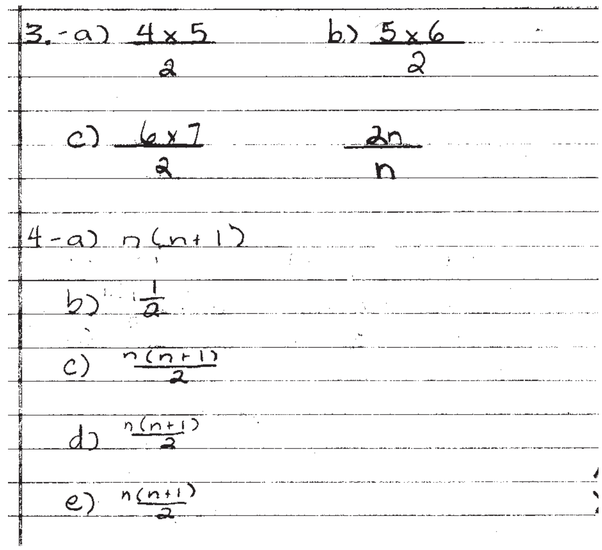


The response to Exercise ❸ contains the correct conjecture regarding the number of squares in a triangular array with *n* squares on each side. Furthermore, the answers to Exercise ❹, indicate that the student is able to use the variable *n* to generalize a pattern. The absence of a response to Exercise ❹ e) suggests that the student has not understood the relationship between the sum of the integers from 1 to *n* and the triangular numbers.

WHAT YOU MIGHT SEE

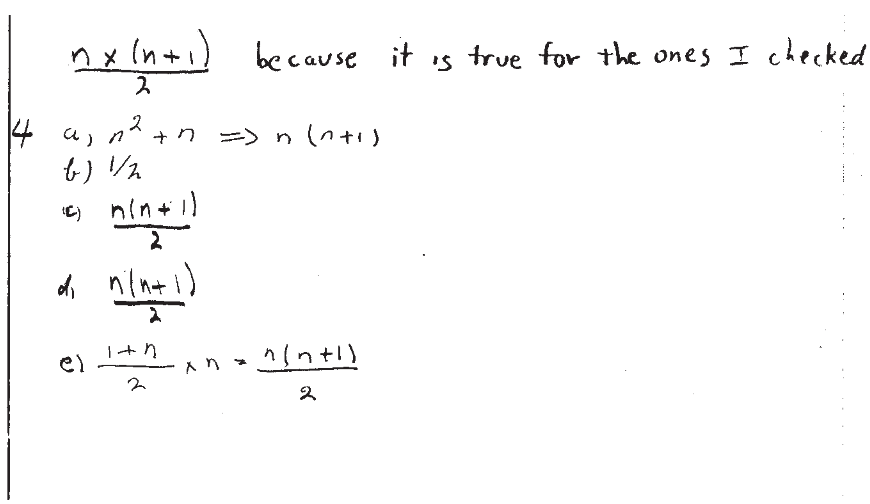
Application of Math Procedures: Translation of Statements into Algebraic Expressions

Level 3



The response to Exercise ❸ contains an incorrect conjecture regarding the number of squares in a triangular array with *n* squares on each side. However, the answers to Exercise ❹ are correct, indicating that the student is able to create the correct algebraic expression in most cases.

Level 4



The response to Exercise ❸ contains the correct conjecture regarding the number of squares in a triangular array with *n* squares on each side. Also the answers to Exercise ❹ are correct, suggesting a high proficiency in writing algebraic expressions within the limits of this activity.

Activity 4 – Teacher Edition

A Formula for the Sum of the Integers from 1 to n

Expectations Addressed

P&A 7-2 extend a pattern, complete a table, and write words to explain the pattern.

P&A 7-3 recognize patterns and use them to make predictions.

P&A 7-4 interpret a variable as a symbol that may be replaced by a given set of numbers.

P&A 7-5 write statements to interpret simple formulas.

P&A 7-6 present solutions to patterning problems and explain the thinking behind the solution process.

P&A 7-7 evaluate simple algebraic expressions by substituting natural numbers for the variables.

Context

In this activity, we share with students one of the fascinating stories from the history of mathematics. Recognized as one of the three greatest mathematicians of all time (together with Archimedes and Newton), Karl Friedrich Gauss showed early promise that later blossomed into an unparalleled mastery of several branches of mathematics. He was also a calculating prodigy and the story told in this lesson shows how he quickly added all the integers from 1 to 100 without algebra.

In the previous lesson, students were left with two challenges. They were to add the integers from 1 to 100 and to add the integers from 1 to 1000. While the first sum can be found in a variety of ways (including mindless addition on a calculator), the second sum requires a more thoughtful process. In particular, it is expected that most students will apply the formula derived in Activity 3 (p. 41) for the n^{th} triangular number to obtain the sum 500 500.

This activity introduces students to Gauss's method and leads them to generalize his method to find a formula for the sum of the integers from 1 to *n*. This provides them with an alternative method for discovering the formula for the n^{th} triangular number. In Exercise ❻, students are encouraged to describe how they use patterns to generalize from specific cases to all cases. Explaining how they use patterns to generalize helps students internalize the concept of a formula as an algebraic recipe for computing. In Exercise ❼, students who have access to graphing calculators are encouraged to create a table of values for the expression $n(n + 1)/2$ to verify that this formula does, in fact, generate the triangular numbers.

When students explore Figurate Numbers on the Internet, they will discover that triangular numbers are a special case of a more general set of polygonal numbers including square and pentagonal numbers.

Activity 4 – Student Page

A Formula For the Sum of the Integers From 1 to n


Karl Friedrich Gauss 1777-1855

In 1787, a ten-year-old boy, named Karl, sat in a classroom in a country that is now called *Germany*. He was doing his arithmetic assignment with a piece of chalk on a small slate. (Why weren't they using ball point pens and paper?) The master teacher had asked the class to add all the integers from 1 to 100. While all the other students were busy arranging the numbers in columns for addition, Karl wrote 5050 on his slate and placed it on the master's desk. The startled master looked in amazement at the correct answer and asked how Karl had obtained the result so quickly. Karl showed him that he had written the numbers from 1 to 50 and then the numbers from 51 to 100 in the reverse direction and then grouped them in pairs as shown here.

1	+	2	+	3	+	… →	+	50
<u>100</u>	+	<u>99</u>	+	<u>98</u>	+	… ←	+	<u>51</u>

Then he asked himself three questions:

"What is the sum of each pair of numbers?"
"How many such pairs are there?"
"What is the sum of *all* the pairs of numbers?"

Once he had answered these three questions, he wrote the number 5050 – the problem had been solved and Karl Friedrich Gauss had given the first hint that he would ultimately become one of the greatest mathematicians who ever lived!

❶. Answer each of the three questions above and show how these answers yield 5050.

❷. Use Gauss's method to find each of these sums.
a) 1 + 2 + 3 + … + 150
b) 1 + 2 + 3 + … + 179
Explain how you modified Gauss's method when there were an odd number of addends.

❸. Another way to find the sum of the integers from 1 to 100 is to take the average of the addends – the mean of the first and last numbers, i.e. (1 + 100)/2 and multiply by the number of addends, i.e. 100. Apply this method to both parts of Exercise ❷, to check your answers.

Activity 4 – Teacher Edition

The Lesson Launch 10 minutes

It is suggested that you launch this lesson by posing the question: *Who do you think was the greatest mathematician of all time?* You may be surprised to discover how few of the great mathematicians are known to your students. The most frequently named are usually those such as Einstein or Newton, who are known to the public through their contributions to science. Ask if anyone has heard of a mathematician named Gauss (rhymes with grouse). Describe for students some of the outstanding contributions of Gauss including his invention of congruence arithmetic and his discovery of non-Euclidean geometry. You might suggest that some students search the Internet for information about Gauss and write up their findings as a project.

After this preliminary discussion, read to the class the story of Gauss given on page 48. Lead the students through the development of Gauss's method by soliciting answers to each of the three questions posed on that page and display the answers on the blackboard or the overhead projector.

Individual Activity 10 minutes

Distribute copies of pages 48 and 49 to all students and ask students to complete Exercises ❷ and ❸. Remind them to show their work. Check that students can correctly apply Gauss's method and the averaging method in Exercise ❸ before proceeding with the cooperative learning activity.

Cooperative Learning Activity 20 minutes

Organize the students in groups of three. Appoint a chair of each group, who is to assign one member the job of recorder, and the other, the job of reporter. The chair is responsible for having the group work through Exercises ❹, ❺, and ❻. The recorder is responsible for recording their answers in a single report. After the group has completed this report, it will be the reporter's responsibility to read the group's answers to these exercises and to demonstrate at the blackboard or on the overhead projector how the 200th triangular number and the sum of the numbers from 1 to 1000 were calculated.

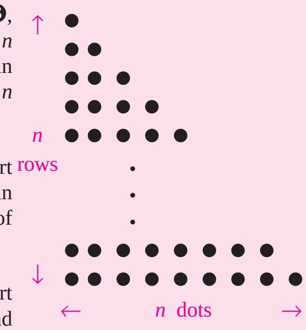
Activity 4 – Student Page

A Formula for the Sum of the Integers from 1 to n

❹. a) Using the method in Exercise ❸, write an algebraic expression in n for the average number of dots in a row for a triangular array with n dots on each side.

b) Multiply your expression in Part a) by the number of rows to obtain an expression for the number of dots in the triangular array.

c) Compare your expression in Part a) with the expression you found in Exercise 4e) of Activity 3 for the sum of the integers from 1 to n. Describe your findings.

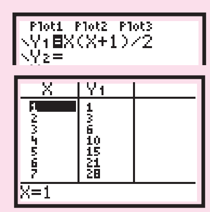


❺. Use the expression you found in Exercise 4 d) to calculate:
a) the 10th triangular number.
b) the 50th triangular number.
c) the 200th triangular number.

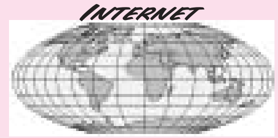
❻. Write a few sentences to explain how you can use patterns in algebra to find the sum of the integers from 1 to 1000 without adding each number. Then write the sum of the integers from 1 to 1000.

❼. If you have a graphing calculator, define $Y_1 = X(X + 1)/2$ as shown. (Note: Calculators such as the TI-73 use X instead of n as a variable.)

Then display the table of values showing the triangular numbers, Y_1. Scroll through the table of values to check your answers to Exercise ❺.



Internet Exploration



Search the Internet using the key words Figurate Numbers.

Report what you discover and give the website addresses of the locations where you found this information.

Here are some websites you may wish to explore on the Internet.
http://www.imsa.edu/edu/math/journal/volume1/webver/triangle.html
http://forum.swarthmore.edu/workshops/usi/pascal/pascal_hexagonal.html

Closure

When all groups have delivered their reports, ask students, *How many dots in a triangular array with n dots on each side?* Students should respond with the formula $n(n + 1)/2$. Ask whether anyone can explain why this formula is true. Usually one or two students are able to offer at least one of these explanations:

- The triangular array is half the number of dots in a rectangular array of n rows and $n + 1$ columns.
- The average number of dots per row is $(n + 1)/2$ and there are n rows.

To extend the concept of triangular numbers, assign each of the cooperative groups the task of completing the Internet exploration on page 49. Ask each group to contribute one printed sheet of information on figurate numbers to be posted on the class project bulletin board.

A Formula for the Sum of the Integers from 1 to n

Karl Friedrich Gauss 1777-1855

THE BETTMANN ARCHIVE

In 1787, a ten-year-old boy named Karl sat in a classroom in a country that is now called *Germany*. He was doing his arithmetic assignment with a piece of chalk on a small slate. (Why weren't they using ballpoint pens and paper?) The master (teacher) had asked the class to add all the integers from 1 to 100. While all the other students were busy arranging the numbers in columns for addition, Karl wrote 5050 on his slate and placed it on the master's desk. The startled master looked in amazement at the correct answer and asked how Karl had obtained the result so quickly. Karl showed him that he had written the numbers from 1 to 50 and then the numbers from 51 to 100 in the reverse direction and then grouped them in pairs as shown here.

1	+	2	+	3	+	… →	+	50
100	+	99	+	98	+	… ←	+	51

Then he asked himself three questions:

"What is the sum of each pair of numbers?"
"How many such pairs are there?"
"What is the sum of *all* the pairs of numbers?"

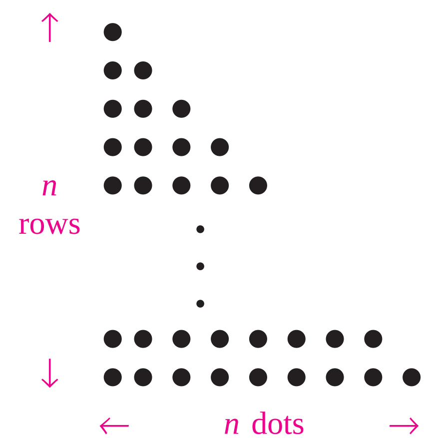
Once he had answered these three questions, he wrote the number 5050 – the problem had been solved and Karl Friedrich Gauss had given the first hint that he would ultimately become one of the greatest mathematicians who ever lived!

❶ Answer each of the three questions above and show how these answers yield 5050.

❷ Use Gauss's method to find each of these sums.
a) 1 + 2 + 3 + … + 150
b) 1 + 2 + 3 + … + 179
Explain how you modified Gauss's method when there were an odd number of addends.

❸ Another way to find the sum of the integers from 1 to 100 is to take the average of the addends. Take the mean of the first and last numbers and multiply by the number of addends. For example, the mean of the first and last numbers is (1 + 100)/2 = 50.5. Multiplying by the number of addends, 100, gives you $50.5 \times 100 = 5050$. Apply this method to both parts of Exercise ❷, to check your answers.

A FORMULA FOR THE SUM OF THE INTEGERS FROM 1 TO n



❹ a) Using the method in Exercise ❸, write an algebraic expression in *n* for the average number of dots in a row for a triangular array with *n* dots on each side.

b) Multiply your expression in ❹ a) by the number of rows to obtain an expression for the number of dots in the triangular array.

c) Compare your expression in ❹ a) with the expression you found in Exercise ❹ e) of Activity 3 for the sum of the integers from 1 to *n*. Describe your findings.

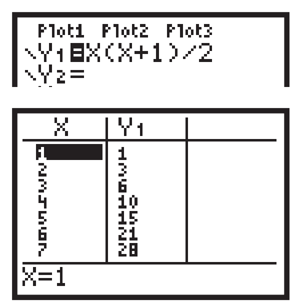
❺ Use your expression from Exercise ❹ d) of Activity 3 to calculate:

a) the 10th triangular number.
b) the 50th triangular number.
c) the 200th triangular number.

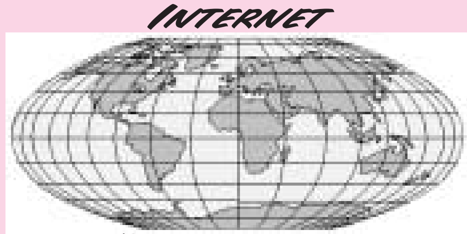
❻ Write a few sentences to explain how you can use patterns in algebra to find the sum of the integers from 1 to 1000 without adding each number. Then write the sum of the integers from 1 to 1000.

❼ If you have a graphing calculator, define $Y_1 = X(X + 1)/2$ as shown. (Note: Calculators such as the TI-73 use X instead of *n* as a variable.)

Then display the table of values showing the triangular numbers, Y_1. Scroll through the table of values to check your answers to Exercise ❺.



INTERNET EXPLORATION



Search the Internet using the key words *Figurate Numbers*.

Report what you discover and give the website addresses of the locations where you found this information.

Here are some other websites you may wish to explore on the Internet.

http://www.imsa.edu/edu/math/journal/volume1/webver/triangle.html
and
http://forum.swarthmore.edu/workshops/usi/pascal/pascal_hexagonal.html

Grade 7

Answer Key for Activity 4

❶ Each pair of numbers has a sum of 101 and there are 50 such pairs, so the sum of *all* the pairs is: $50 \times 101 = 5050$.

❷ a) $1 + 2 + 3 + \ldots + 150$ can be arranged in 75 pairs:

1	+	2	+	3	+	→ …	+	75
<u>150</u>	+	<u>149</u>	+	<u>148</u>	+	…	+	<u>76</u>
151		151		151		←		151

The sum is $75 \times 151 = 11\,325$.

b) There is an odd number of addends, so we sum the numbers from 1 to 178 and add 179 later. As in ❹ a), we group the numbers in pairs and observe there are 89 pairs and each pair has a sum 180.

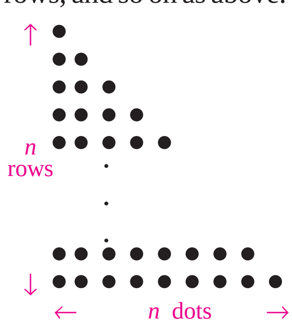
1	+	2	+	3	+	…	+	89
<u>178</u>	+	<u>177</u>	+	<u>176</u>	+	…	+	<u>90</u>
179		179		179				179

The sum is therefore $89 \times 179 + 179 = 16\,110$.

❸ a) We observe that all pairs of numbers shown in Exercise ❷ a) have a mean of 151/2. Since there are 150 numbers with a mean of 151/2, the total of all such numbers is $150 \times 151/2 = 11\,325$.

b) We observe that the middle number is 90 and all pairs of numbers shown in Exercise ❷ b) have a mean of 90. Therefore there are 179 numbers with a mean of 90, and the total of all such numbers is $179 \times 90 = 16\,110$.

❹ a) We pair the first and last rows, and so on as above. The average number of dots per row is $(n + 1)/2$.



b) There are n rows with an average of $(n + 1)/2$ dots per row, so the total number of dots is given by $n(n + 1)/2$.

c) The expression in ❹ b) is the same as the expression found in Exercise ❹ e) of Activity 3.

❺ Substituting the values $n = 10$, 50, and 200 into the expression for triangular numbers $n(n + 1)/2$, we obtain:

a) 10^{th} triangular number 55.

b) 50^{th} triangular number 1275.

c) 200^{th} triangular number 20 100.

❻ Answers will vary. Students may describe the method of pairings used by Gauss, the method of averages or the use of arrays of dots. The real power of algebra comes from the extension of these patterns to the general case. Upon discovering the expression $n(n + 1)/2$, we can compute the sum of the positive integers from 1 to n for any value of n.

The sum of the integers from 1 to 1000 is 500 500.

❼ As an extension to the graphing calculator activity with triangular numbers, have students graph the function X(X + 1)/2, in various windows. Discuss how the triangular numbers increase in non-linear fashion. Then have students trace along the graph to find the n^{th} triangular number for various values of n.

Internet Exploration

The first website listed provides a brief derivation of the formula for triangular numbers and then provides some explorations that require students to find patterns in a variety of different sequences. The second reference provides the students with additional opportunities for exploration. For a little history on Pythagoras and triangular numbers, refer your students to:

http://mathserv.math.sfu.ca/history_of_math/Europe/Euclid/Euclid3000BC/pythagoras.html

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

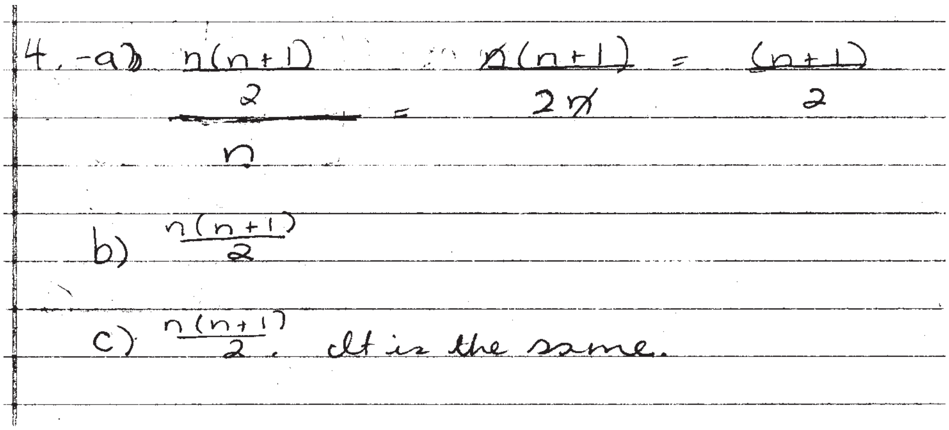
Scoring Guide for Activity 4

	Level 1	Level 2	Level 3	Level 4
APPLICATION **Translation of Statements into Algebraic Expressions** (exercises ❹ a) & b)) P&A 7-8	•Gives incorrect or no algebraic expression in most instances.	•Gives incorrect or no algebraic expression in several instances.	•Gives correct algebraic expression in most instances.	•Gives correct algebraic expression in almost all instances.
APPLICATION **Evaluation of Algebraic Expressions by Substitution** (exercises ❺ & ❻) P&A 7-7	•Major errors and/or omissions are evident in the evaluation of algebraic expressions.	•Several minor errors and/or omissions are evident in the evaluation of algebraic expressions.	•A few minor errors and/or omissions are evident in the evaluation of algebraic expressions.	•Almost no errors are evident in the evaluation of algebraic expressions.
COMMUNICATION **Articulation of a Mathematical Relationship or Procedure.** (exercises ❷, ❹ c) & ❻) P&A 7-1, 7-2 & 7-6	•Explanations are either missing or lack clarity. •Appropriate terminology is rarely used.	•Explanations are usually clear, but they lack precision. •Some appropriate terminology is used.	•Most explanations are clear and precise. •Appropriate terminology is often used.	•Almost all explanations are clear and precise and appropriate terms and symbols are used.

WHAT YOU MIGHT SEE

Application: Translation of Statements into Algebraic Expressions

Level 4



4\. a) $\dfrac{\frac{n(n+1)}{2}}{n} = \dfrac{n(n+1)}{2n} = \dfrac{(n+1)}{2}$

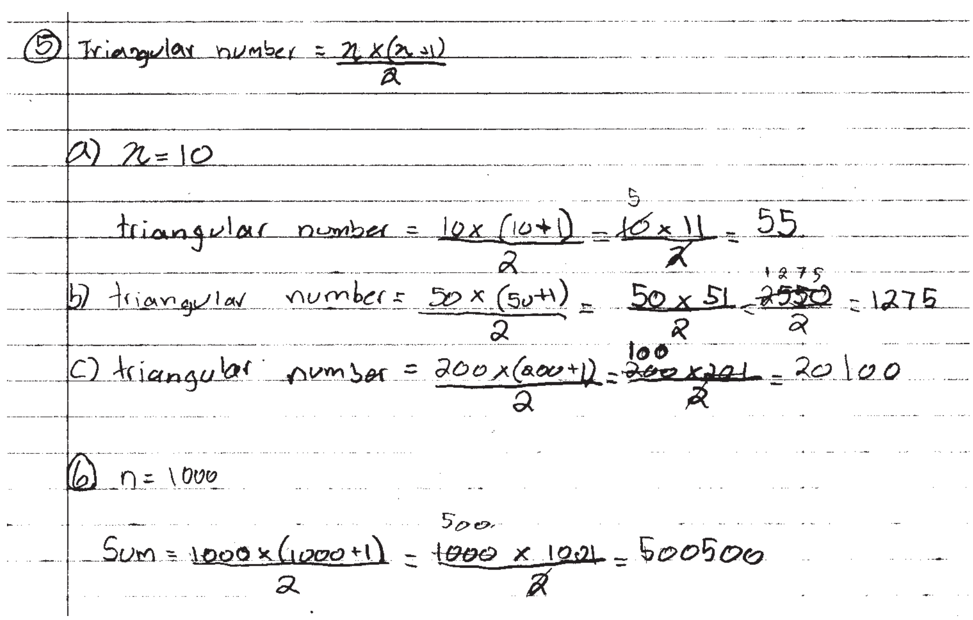
b) $\dfrac{n(n+1)}{2}$

c) $\dfrac{n(n+1)}{2}$. It is the same.

In the response to Exercise ❹ a), this student has written the *total* number of dots in the $n \times n$ triangular array as $n(n + 1)/2$ and then divided by the number of rows n to obtain the average number of dots $(n + 1)/2$. In ❹ b) the student multiplies by n to obtain the formula for the n^{th} triangular number and in ❹ c), correctly identifies it as the same as the formula for the sum of the integers from 1 to n. The correct algebraic expressions were given in all cases.

Application: Evaluation of Algebraic Expressions By Substitution

Level 4



⑤ Triangular number $= \dfrac{n \times (n+1)}{2}$

a) $n = 10$

triangular number $= \dfrac{10 \times (10+1)}{2} = \dfrac{10 \times 11}{2} = 55$

b) triangular number $= \dfrac{50 \times (50+1)}{2} = \dfrac{50 \times 51}{2} = \dfrac{2550}{2} = 1275$

c) triangular number $= \dfrac{200 \times (200+1)}{2} = \dfrac{200 \times 201}{2} = 20100$

⑥ $n = 1000$

Sum $= \dfrac{1000 \times (1000+1)}{2} = \dfrac{1000 \times 1001}{2} = 500500$

In the responses to Exercises ❺ and ❻, the student substituted the appropriate values in all cases and evaluated the expressions to obtain the correct answers. The statements were clearly written and the computations undertaken were evident.

WHAT YOU MIGHT SEE

Communication: Articulation of a Mathematical Relationship or Procedure

Level 1

Activity #4

1) Sum-101 50-pairs sum-5050

2a) 151 x 75 = 1832 b) 1954 left middle number out and added it later.

4a) - I don't comprehend

This student response reveals some serious difficulties. In the response to Exercise ❷ a), the student has the correct expression 151 × 75, but applies the multiplication algorithm incorrectly and obtains the product 1832, which is clearly the wrong order of magnitude. A similar mistake in Exercise ❷ b), confirms that this student has fundamental difficulty with multiplication. The statement, “left middle number out and added it later”, shows a valid strategy for adding an odd number of addends, and suggests that the student has some skills in articulating mathematical procedures. The student’s response “don’t comprehend” in Exercise ❹ a), is a clear indication that the student does not understand the algebraic representation of triangular numbers and requires help.

Level 3

1. a) – 101
   – 50 pairs
   – sum of all the pairs is 5050

2. a) 11,325

   b) 16,110
   If there is an odd number of addends, then leave out the largest number and then add up all the pairs. After adding up all the pairs, add on the last number.

The responses shown above are typical of this student’s responses throughout this activity. The explanations were usually clear and concise, albeit brief. Appropriate terminology was often used and most of the explanations were complete.

WHAT YOU MIGHT SEE

Communication: Articulation of a Mathematical Relationship or Procedure

Level 4

2\. a) i) The sum of each pair all equal to 151.

ii) There are 75 pairs.

iii) The sum of all the pairs is 11325.

b) i) The sum of each pair all equal to 180.

ii) There are 89.5 pairs

iii) The sum of all the pairs is 16110.

(Whenever there is a odd number, the pairs come out as decimal numbers. In this case, I left the decimated number and carried on. In a case where the answer should come out as a decimated number, you may want to round up to the next place value)

6\. Using either the method on question #3 or Gauss's method, you use the formula N(N+1)/2 or use the three questions Gauss had used.

For example, 1 + 2 + 3 ... + 1000 = ?

$\frac{N(N+1)}{2} = \frac{\overset{500}{\cancel{1000}}(1000+1)}{\cancel{2}\,1} = 500 \times 1001$

= 500,500

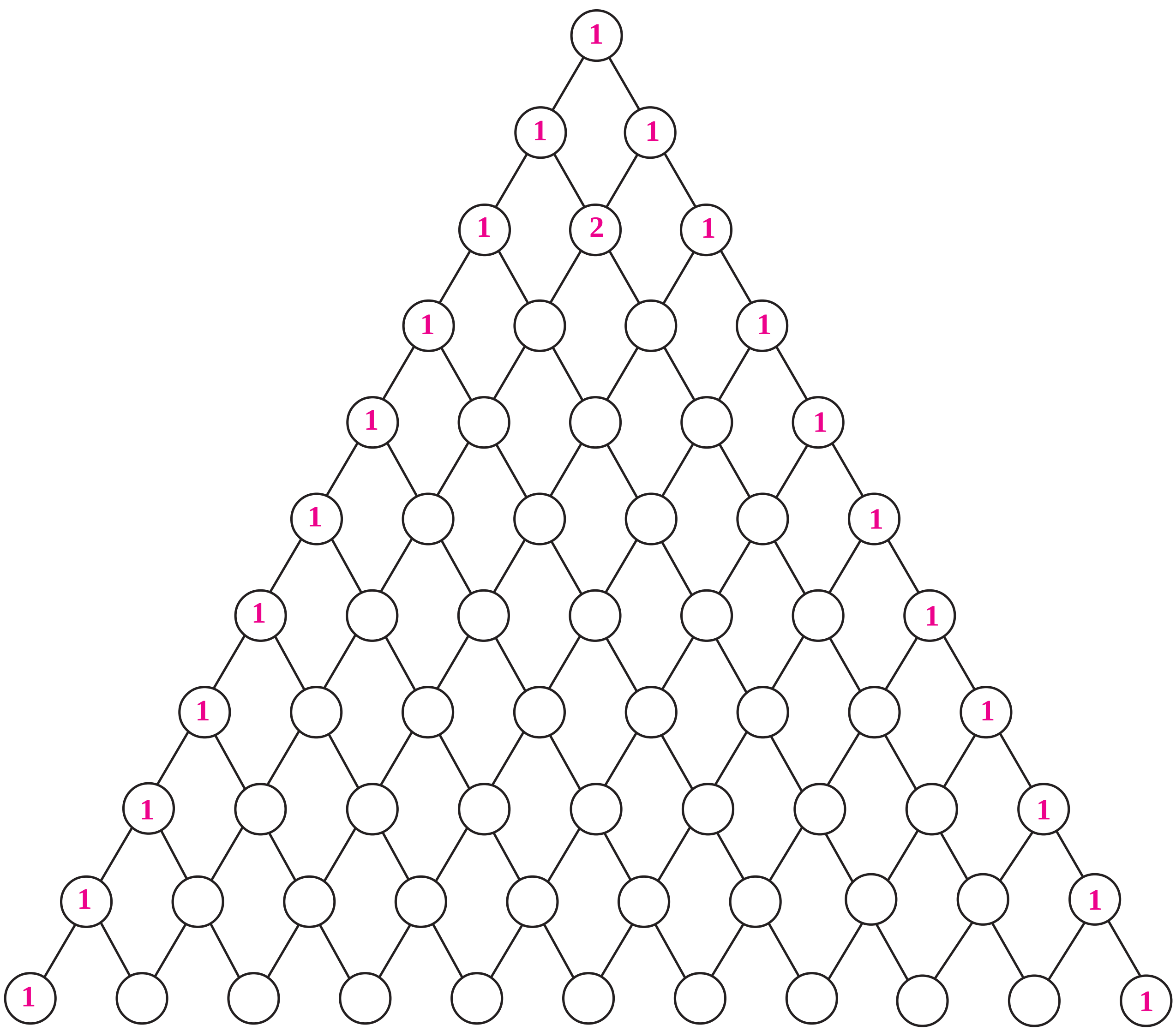
In the response to Exercise ❷, this student has modified Gauss's method to accommodate the situation where there is an odd number of addends, by treating the 179 addends as 89.5 "pairs." The student uses this approach to obtain the correct sum, and then expands the idea even further by inventing a new term, "decimated numbers" (rather than using the conventional term, "decimal numbers.") In response to Exercise ❻, the student offers three different methods for adding the numbers from 1 to 1000 and executes correctly the substitution into the formula to obtain the correct result. This student's work is exemplary.

A Template For Pascal's Triangle

Blaise Pascale 1623-1662

THE BETTMANN ARCHIVE

Pascal was a mathematical genius who at the age of 18 invented a mechanical calculating machine – about 300 years before the development of modern calculators! He was also a cofounder (with Pierre Fermat) of the Theory of Probability. In his development of probability, he used the properties of the triangle below. Although Pascal had not discovered this triangle, it was named *Pascal's triangle* because he discovered so many of its special properties. His friend Pierre Fermat, who was also a great mathematician, tried to get Pascal interested in Number Theory, so he sent him a note stating that every integer can be written as a sum of either one or two or three triangular numbers. For example, 9 = 6 + 3 and 21 = 10 + 10 + 1 or 21 = 15 + 6. Do you think this is a true statement? Check it out.



impact math

focus on the future

PART IV

Algebra in Grade 8

Patterning & Algebra: Grade 8

Overall Expectations

By the end of Grade 8, students will:

- identify the relationships between whole numbers and variables;
- identify, create, and discuss patterns in algebraic terms;
- evaluate algebraic expressions;
- identify, create, and solve simple algebraic equations;
- apply and defend patterning strategies in problem solving situations.

Specific Expectations

(For convenient reference, the specific expectations are coded. For example, P&A 8-3 denotes the third Patterning and Algebra expectation in Grade 8.)

Students will:

Modelling

P&A 8-1 – describe and justify a rule in a pattern;

P&A 8-2 – write an algebraic expression for the n^{th} term of a numeric sequence;

P&A 8-3 – find patterns and describe them using words and algebraic expressions;

P&A 8-4 – use the concept of variable to write equations and algebraic expressions;

P&A 8-5 – investigate inequalities and test whether they are true or false by substituting whole number values for the variables (e.g., in $4x \geq 18$, find the whole number values for x);

P&A 8-6 – write statements to interpret simple equations;

P&A 8-7 – present solutions to patterning problems and explain the thinking behind the solution process;

Linear Equations

P&A 8-8 – evaluate simple algebraic expressions, with up to three terms, by substituting fractions and decimals for the variables;

P&A 8-9 – translate complex statements into algebraic expressions or equations;

P&A 8-10 – solve and verify first-degree equations with one variable, using various techniques involving whole numbers and decimals;

P&A 8-11 – create problems giving rise to first-degree equations with one variable and solve them by inspection or by systematic trial;

P&A 8-12 – interpret the solution of a given equation as a specific number value that makes the equation true.

Activity 1 – Teacher Edition

Where Should They Hold the Fundraising Party?

Expectations Addressed

P&A 8-4 use the concept of variable to write equations and algebraic expressions.

P&A 8-8 evaluate simple algebraic expressions, with up to three terms, by substituting fractions and decimals for the variables.

P&A 8-9 translate complex statements into algebraic expressions or equations.

Context

In this activity, Jennifer and Steve (Grade 8 students) and their committee are organizing a fundraising party. They contact three different locations to ascertain the price of renting a hall and providing snacks. The quotes they receive can be represented by these linear functions, where n denotes the number of guests.

Galaxy Inn

$$\text{Cost in dollars} = \begin{cases} 400 & n \le 23 \\ 17n & n > 23 \end{cases}$$

Noble Pines Country Club

Cost in dollars = $14n + 90$

Holiday Lodge

$$\text{Cost in dollars} = \begin{cases} 500 & n \le 25 \\ 500 + 12(n-25) & n > 25 \end{cases}$$

(The functions look quite formidable in this form, so students are asked only to write the expressions for values of n that are suitably large.) Activity 1 deals only with the quote from the Galaxy Inn. Students are asked to explain the meaning of a "minimum charge" and to write the linear expression $17n$ to represent the cost for $n > 23$. They form a table of values for this linear expression and then graph it for values of n that are multiples of 10 between 20 and 70.

Activity 1 – Student Page

Where Should They Hold the Fundraising Party?

Jennifer, Steve, and their committee are planning a fundraising event to raise money for charity. They want to rent a hall and provide food and entertainment. Everyone in attendance will pay an admission fee to cover costs and the profit will be contributed to a local charity. To minimize the cost of the hall, Jennifer and Steve call three establishments to obtain quotes. In this unit, you will analyze the three quotes that they receive and answer the question, *Where should they hold the fundraising party?*

Discuss

❶ Explain what is meant by "a minimum total charge of $400".

Activity 2 explores the linear function $14n + 90$ associated with quote from Noble Pines C. C., and compares this quote with that of the Galaxy Inn. By investigating the value of n at which both quotes yield the same cost, students explore the solution of the equation $17n = 14n + 90$. Activity 3, in a similar way, compares the linear functions $14n + 90$ and $500 + 12(n – 25)$ and helps students find the value of n for which costs are equal. Finally, Activity 4 consolidates all the previous work, requiring students to express, as inequalities, the ranges of n for which each quote is optimal, and on that basis choose a site for the party.

Activity 1 – Teacher Edition

The Lesson Launch 5 minutes

Set the context for this lesson by asking a series of questions such as:

- *If you had to organize a fundraising party at some location, what arrangements would you have to make?*
- *How would you choose the location for the party?*
- *What factors would you consider? Why?*

Distribute copies of student pages 62-63. Select three students in turn to read each of the quotes presented. Discuss the meanings of a “minimum charge” and a “flat fee.” Ensure that students know the difference between the two. Ask the students to conjecture which of the quotes they think is the best, and why.

Initiating Activity 20 minutes

Have students work individually to complete Exercises ❶ to ❺ and record their answers in their notebooks. As you walk around the room, you will be able to determine which students are having difficulty with the terminology or the concept of the variable *n*. Run off copies of page 18 and encourage students who finish Exercises ❶ to ❺ first to work on problem 1 while the other students complete these exercises. (See p. 93 for solutions to the problems on p. 18.) When almost all students have completed these exercises, discuss the answers to questions ❶ through ❺, to ensure the students are prepared to graph the linear function 17*n*.

Paired Activity 15 minutes

Group students in pairs and provide them with squared paper (see template p. 55). Have students complete Exercise ❻ on page 63, ensuring both students in each pair make their own graphs, taping them in their notebooks for future use.

If students have access to a TI-73, TI-83, or CASIO *fx-7400G* graphing calculator, ask them to define Y = 17X and display a table of values for positive integral values of X. Have them scroll through the table to check their answers to Exercises ❺ c) and ❻ c). Ask them to graph this function and compare it with the graph they created for Exercise ❻.

Activity 1 – Student Page

Computing Costs at the Galaxy Inn

All the exercises in this activity pertain to the quote from the Galaxy Inn.

❷ How much would the Galaxy Inn charge for:
a) 10 guests? b) 20 guests? c) 30 guests?

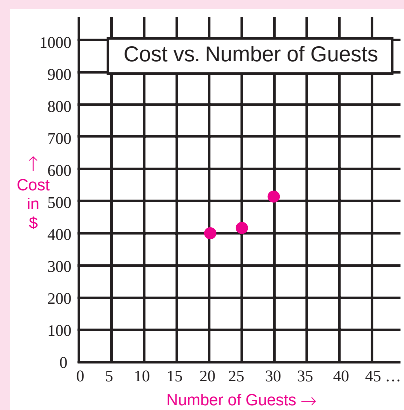
❸ a) How many guests would incur a total cost of $901?
b) What is the largest number of guests that can be accommodated for $1000?
c) How many guests would it take to achieve an average cost of $17 per guest?

❹ a) Make a table showing the cost for each number of guests between 25 and 35.
b) Use your table to check your answer to Exercise ❷ c)

Number of Guests	Cost
25	
26	
.	
.	
.	
34	
35	

❺ a) Describe how you could compute the cost for any given number of guests (greater than 23).
b) Let *n* represent the number of guests attending the party. Write an expression for the cost for *n* guests (where $n > 23$).
c) Use your expression in ❺ b) to calculate the cost for:
(i) 45 guests. (ii) 55 guests. (iii) 65 guests.
d) Use your expression in ❺ b) to calculate how many guests would cost: (i) $561. (ii) $884. (iii) $1156.

❻ a) Using a full sheet of squared paper, make a graph showing the cost of *n* guests for these values of *n*: 20, 25, 30, 40, 50, 60, 70. Save your graph. You will need it in the next activities.



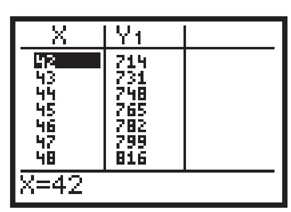
b) Join the dots in your graph. Write a sentence to describe any pattern in your graph.

c) Use your graph in ❻ b) to estimate the cost for:
(i) 45 guests. (ii) 55 guests. (iii) 65 guests.
Compare these answers with your answers in ❺ c).

d) Use your expression in ❻ b) to estimate how many guests would cost:
(i) $561. (ii) $884. (iii) $1156.
Compare these answers with your answers in ❺ d).

Closure

When all the students have finished, invite a pair to display their graph and explain how they estimated the cost for 45 guests. Ask, *Should the points on the graph be joined by a straight line? Why or Why not?* Ask whether the cost function has any meaning for values of *n* such as *n* = 28.5. If you have a graphing calculator viewscreen, display a table of values for Y = 17X and show how we can scroll through a table of values to check answers. Then graph this function on the screen and trace along it to show its relationship to the table.



At the end of class, distribute problem 1 (p. 18) to all students and assign it as a homework challenge for those who believe they are good at logic. Invite them to involve their parents.

Where Should They Hold the Fundraising Party?

Jennifer, Steve, and their committee are planning a fundraising event to raise money for charity. They want to rent a hall and provide food and entertainment. Everyone in attendance will pay an admission fee to cover costs and the profit will be contributed to a local charity. To minimize the cost of the hall, Jennifer and Steve call three establishments to obtain quotes. In this unit, you will analyze the three quotes that they receive and answer the question, *Where should they hold the fundraising party?*

Discuss

❶ Explain what is meant by “a minimum total charge of $400.”

WHERE SHOULD THEY HOLD THE FUNDRAISING PARTY?

Computing Costs at the Galaxy Inn

All the exercises below pertain to the quote from the Galaxy Inn.

❷ How much would the Galaxy Inn charge for:
a) 10 guests? b) 20 guests? c) 30 guests?

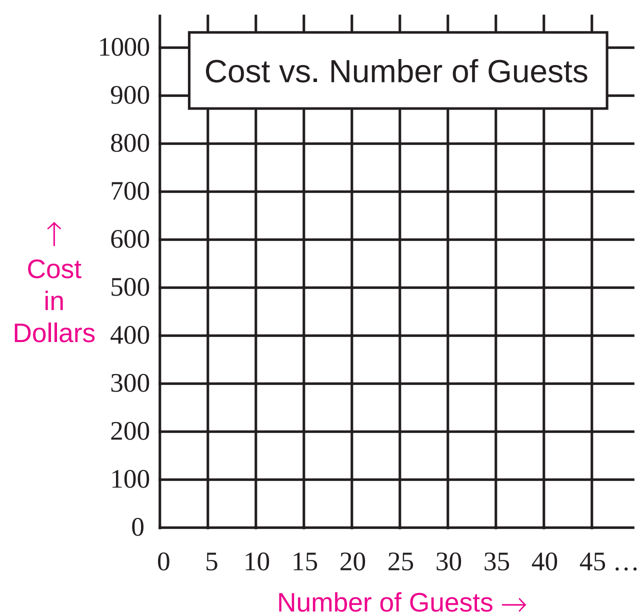
❸ a) How many guests would incur a total cost of $901?
b) What is the largest number of guests that can be accommodated for $1000?
c) How many guests would it take to achieve an average cost of $17 per guest?

❹ a) Make a table showing the cost for each number of guests between 25 and 35.
b) Use your table to check your answer to Exercise ❷ c).

Number of Guests	Cost
25	
26	
•	
•	
•	
34	
35	

❺ a) Describe how you could compute the cost for any given number of guests (greater than 23).
b) Let n represent the number of guests attending the party. Write an expression for the cost for n guests (where $n > 23$).
c) Use your expression in ❺ b) to calculate the cost for:
(i) 45 guests. (ii) 55 guests. (iii) 65 guests.
d) Use your expression in ❺ b) to calculate how many guests would cost:
(i) $561. (ii) $884. (iii) $1156.

❻ a) Using a full sheet of squared paper, make a graph showing the cost of n guests for these values of n: 20, 25, 30, 40, 50, 60, 70. Save your graph. You will need it in the next activities.
b) Join the dots in your graph. Write a sentence to describe any pattern in your graph.
c) Use your graph in ❻ b) to estimate the cost for:
(i) 45 guests. (ii) 55 guests. (iii) 65 guests.
Compare these answers with ❺ c).
d) Use your graph in ❻ b) to estimate how many guests would cost:
(i) $561. (ii) $884. (iii) $1156.
Compare these answers with ❺ d).



Grade 8
Answer Key for Activity 1

❶ A "minimum total charge of $400" means that no matter how many guests there are, the total charge will not be less than $400.

❷ a) Cost for 10 guests would be $400.
b) Cost for 20 guests would be $400.
c) Cost for 30 guests would be $17 \times 30 = \$510$.

❸ a) $901 \div 17 = 53$, so 53 guests would cost $901.
b) $1000 \div 17 = 58.82\ldots$, so $1000 is enough to pay for 58 guests but not for 59 or more.
c) The cost at the Galaxy Inn is $17 per person when the total cost is at least $400. This occurs when the number of guests is greater than or equal to $400 \div 17$; i.e. when the number of guests is 24 or more.

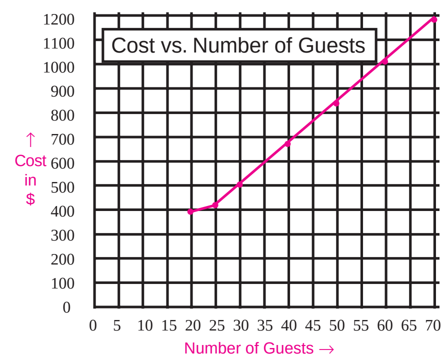
❹ a) The table is shown here.

Number of Guests	Cost
25	425
26	442
27	459
28	476
29	493
30	510
31	527
32	544
33	561
34	578
35	595

b) We observe from the table that the cost corresponding to 30 guests is $510. This verifies our answer to ❷ c).

❺ a) Multiply the number of guests by 17 to obtain the cost in dollars.
b) Cost in dollars = $17n$ for $n > 23$
c) (i) $17 \times 45 = \$765$
(ii) $17 \times 55 = \$935$
(ii) $17 \times 65 = \$1105$
d) (i) 33 guests (ii) 52 guests (iii) 68 guests

❻ a) The graph should look like this.



Note: The cost for 23 guests is $400, for 24 guests it is $408, and for 25 guests it is $425 and it increases from there at $17 per guest. Whether the students graph the cost for $n = 20, 25, 30 \ldots$ or for $n = 20, 21, 22, 23, 24, 25, \ldots$ the graph will have two bends in it.

b) The dots for $n \geq 25$ lie along a straight line.
c) The students' estimates should be close to $765, $935, and $1105 respectively, as obtained in Exercise ❺ c).
d) The students' estimates should be close to 33 guests, 52 guests, and 68 guests respectively, as in Exercise ❺ d).

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

CONCEPTS	Level 1	Level 2	Level 3	Level 4
Representation of a Linear Function by: **• a table** **• a linear equation** **• a graph** (exercises ❹ – ❻) P&A 8-4, 8-8, 8-9, 8-10	•Two or more of the three representations (table of values, equation, or graph) of the Galaxy Inn cost function are missing or incorrect.	•One of the three representations (table of values, equation, or graph) for the Galaxy Inn cost function is missing or incorrect.	•The correct table of values, equation, and graph are given for the Galaxy Inn cost function, with a few minor errors.	•The correct table of values, equation, and graph are given for the Galaxy Inn cost function, with almost no errors.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

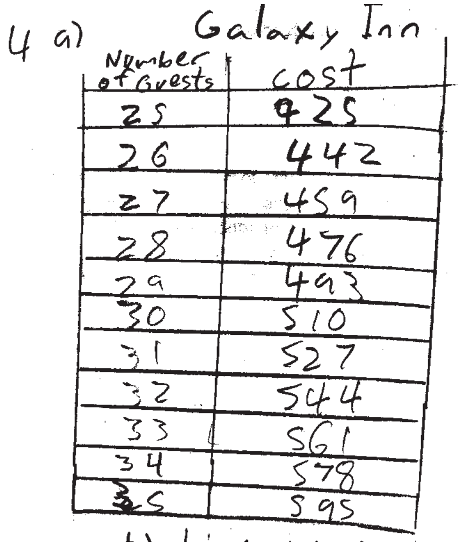
Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

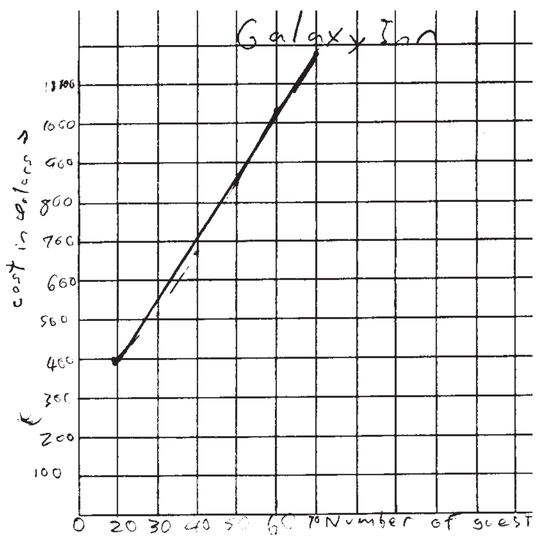
Representation of a Linear Function by a Table, an Equation & a Graph

Level 1



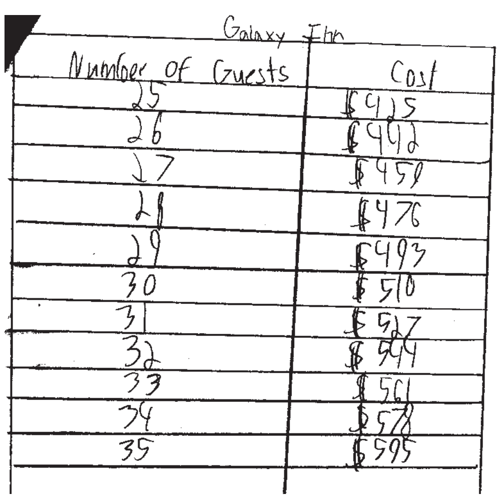
4 a) Galaxy Inn

Number of Guests	cost
25	425
26	442
27	459
28	476
29	493
30	510
31	527
32	544
33	561
34	578
35	595



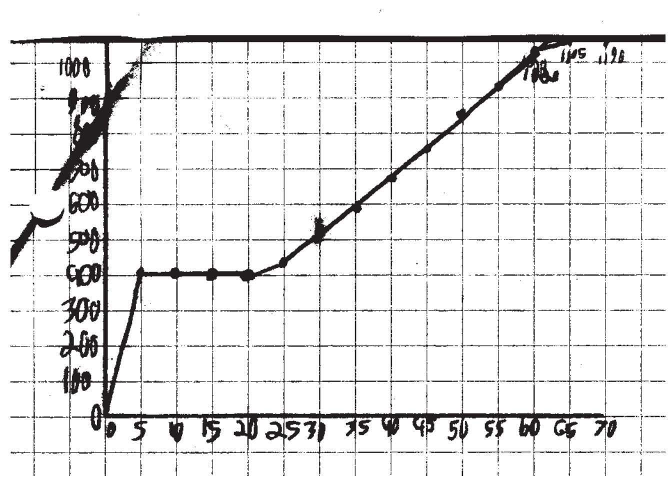
The student has completed the table of values correctly. The graph shows no understanding of the condition that there is a minimum charge of $400, and shows a charge of $400 beginning at 20 guests. Also the student was unable to write the cost function 17*n* in Exercise ❺ b). Only one of the three representations of the Galaxy Inn cost function was completed correctly.

Level 2



Galaxy Inn

Number of Guests	Cost
25	$425
26	$442
27	$459
28	$476
29	$493
30	$510
31	$527
32	$544
33	$561
34	$578
35	$595



The student has completed the table of values correctly. The graph shows an understanding of the condition that there is a minimum charge of $400, however, the graph is incorrect in the range between 0 and 5 guests, and shows a charge of $400 beginning at 20 guests. Furthermore the axes of the graph are not labelled. In addition, the student was unable to write the cost function 17*n* in Exercise ❺ b). Two of the three representations of the Galaxy Inn cost function were completed, albeit with some minor errors.

WHAT YOU MIGHT SEE

REPRESENTATION OF A LINEAR FUNCTION BY A TABLE, AN EQUATION & A GRAPH

Level 3

Activity 1

4a)

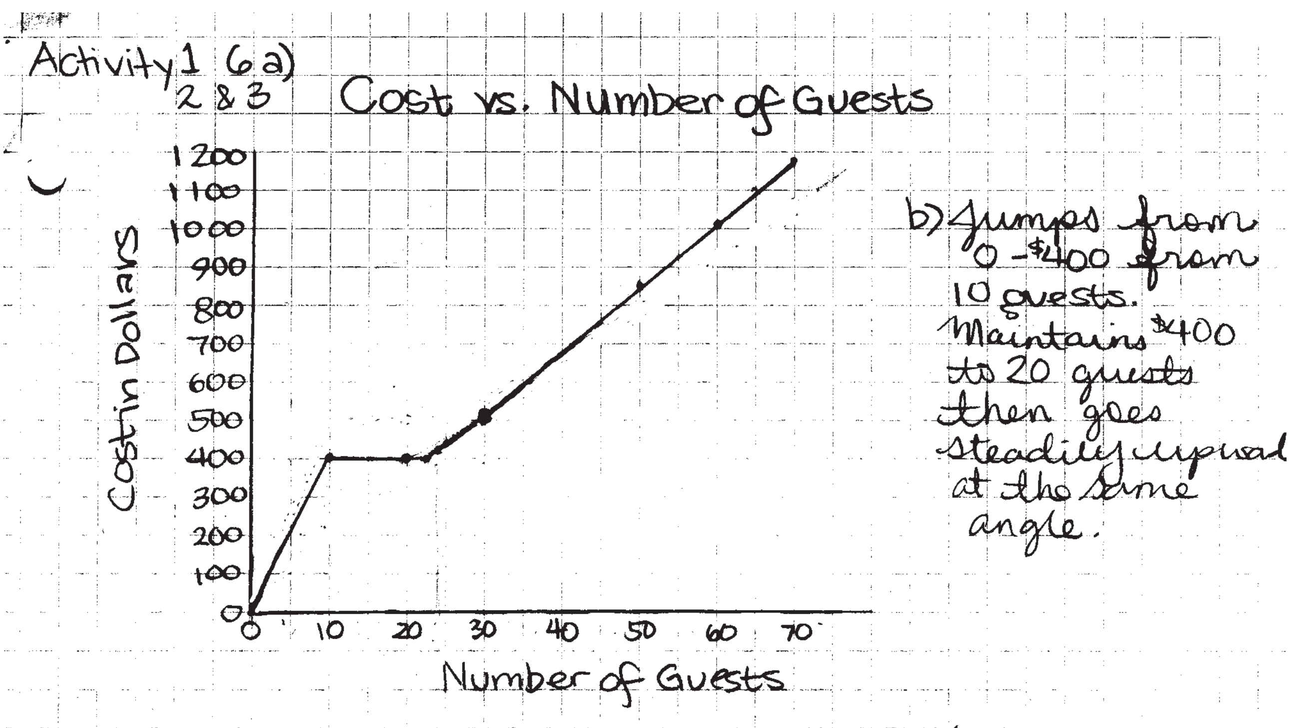
# of Guests	Cost ($)
25	$425
26	$442
27	$459
28	$476
29	$493
30	$510
31	$527
32	$544
33	$561
34	$578
35	$595

Galaxy Inn

❺. a) Describe how you could compute the cost for any given number of guests (greater than 23). n = number of guests 17n 17×n

b) Let n represent the number of guests attending the party. Write an expression for the cost for n guests (where $n > 23$). 17n

↑ The student wrote the answers to this exercise on the sheet.



The student has completed the table of values correctly. The graph shows an understanding of the condition that there is a minimum charge of $400, however, the graph is incorrect in the range between 0 and 10 guests, and shows a charge of $400 beginning at 10 guests. It also begins to increase beyond $400 at 24 guests. The student was able to write the cost function 17n in Exercise ❺ b) as shown above. All three of the representations of the Galaxy Inn cost function were completed, albeit with a few minor errors.

Activity 2 – Teacher Edition

Comparing the Galaxy Inn with the Noble Pines Country Club

Expectations Addressed

P&A 8-4 use the concept of variable to write equations and algebraic expressions.

P&A 8-9 translate complex statements into algebraic expressions or equations.

P&A 8-10 solve and verify first-degree equations with one variable, using various techniques involving whole numbers and decimals.

P&A 8-12 interpret the solution of a given equation as a specific number value that makes the equation true.

Context

This activity extends the investigation of linear functions in Activity 1 from the exploration of the cost function $17n$, where n denotes the number of guests, to the exploration of the cost function $14n + 90$. After students explain in words how to compute the costs for various numbers of guests at the Noble Pines Country Club, they are asked in Exercise ❺b) to translate this verbal description into an algebraic expression (see Expectation 8-9). In Exercise ❻, students graph the linear function $14n + 90$ on the grid used in Activity 1 for the function $17n$. This enables them to compare the graphs of the cost functions of the Galaxy Inn and the Noble Pines Country Club. In Exercise ❻ d), students are asked to indicate the value of n at which the graphs cross. They are expected to observe that the graphs cross at $n = 30$, indicating that the costs for both sites are equal for this number of guests. This is tantamount to asking for the graphical solution of the linear equation $17n = 14n + 90$, and speaks directly to expectation 8-12. A formal algebraic solution of this equation is not expected at this point.

The purpose of Activity 2 is to help students create and interpret the tabular and the graphical representations of linear functions. They will observe that adding a flat fee makes that option a more expensive proposition for small values of n, but for a sufficiently large value of n, the per person cost (coefficient of n) is more important. Another objective of this lesson is to help students use inequalities to determine the range in the values of n for which each option is preferred. The cartoon on page 70 is intended to clarify for students the meaning of the "best option." That is, the least expensive option is that which renders the lowest total cost, and this is the same as the lowest cost per guest.

Activity 2 – Student Page

Comparing The Galaxy Inn With The Noble Pines Country Club

ACTIVITY 2 – TEACHER EDITION

The Lesson Launch 15 minutes

Before launching this lesson, ask whether anyone was successful in solving the problem handed out at the end of the previous class (problem 1, p. 18). Display a copy of the problem on the overhead projector and discuss what it means when a two-pan scale is in balance. Ask students,

- *Give an algebraic expression for the total mass of the coins on the left pan; on the right pan.*
- *The two pans are in balance; what does that tell us?*
- *How would you state this algebraically?*
- *Finding the value of n that makes both sides equal is called "solving for n"; how could we do this?*

Allow the students to struggle with the last question before providing a hint such as, *How much greater in mass are the n coins on the left pan than the n coins on the right pan?* and *If the n coins on the left pan are 3n grams more than the n coins on the right pan, then why are the pans in balance?* Explain to students that they will be solving the problem in a different way as they work through this activity.

To launch the lesson, distribute student pages 70 and 71 and have students read to themselves the cartoon on page 70. Then ask students whether the boy is using an appropriate method for determining the "best deal." Ask why or why not.

Paired Activity 30 minutes

Group the students in pairs. Have each pair complete Exercises ❶ through ❺ with each student recording the answers in his notebook. Ask students, when finished, to complete Exercise ❻ on the grid they used in Activity 1. When almost all students are finished, ask about the value of *n* at which the graphs cross. Ask what this means about the costs of the two different sites. Ensure that students understand not only that this is the value of *n* for which the two sites have the same cost, but also that as the number of guests is increased beyond this, the Noble Pines Country Club offers the lower cost.

If students have access to a TI-73, TI-83, or CASIO *fx-7400G* graphing calculator, ask them to define $Y_1 = 17X$ and $Y_2 = 14X + 90$ and display a table of values showing Y_1 and Y_2 as functions of X. Have them scroll through the table to check their answers to Exercises ❺ c) and ❻ d). Ask them to graph these functions and trace along the graphs to check their answers to Exercise ❻ d).

ACTIVITY 2 – STUDENT PAGE

COMPARING THE GALAXY INN WITH THE NOBLE PINES COUNTRY CLUB

All the exercises in this activity pertain to the quote from the Noble Pines Country Club.

❶ Explain what is meant by "a flat fee of \$90". How is this different from a minimum charge?

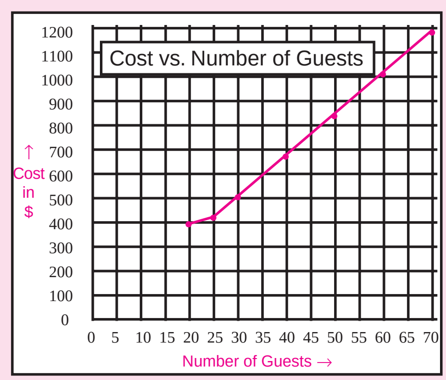
❷ How much would the Noble Pines Country Club charge for:
a) 10 guests? b) 20 guests? c) 30 guests?

❸ a) How many guests would incur a total cost of \$566?
b) What is the largest number of guests that can be accommodated for \$1000?
c) How many guests would it take to achieve an average cost of \$17 per guest?

❹ a) Make a table showing the cost for each number of guests between 25 and 35.
b) Use your table to check your answer to Exercise ❸ a).
c) Compare your table with the table you constructed in Exercise ❹ of Activity 1. How many guests are needed to make Noble Pines cost per guest less than the Galaxy Inn?

❺ a) Describe how you could compute the cost for any given number of guests.
b) Let *n* represent the number of guests attending the party. Write an expression for the cost for *n* guests.
c) Use your expression in ❺ b) to calculate the cost for:
(i) 40 guests. (ii) 50 guests. (iii) 55 guests.
d) Use your expression in ❺ b) to calculate how many guests would cost: (i) \$692. (ii) \$888. (iii) \$1042.

❻ The graph you constructed in Activity 1 should look like this.



a) Using the same grid, plot points in a different colour to show the cost at Noble Pines for *n* guests for these values of *n*: 20, 25, 30, 40, 50, 60, 70. Join the new dots in your graph.
b) Use your graph to check your answers to check your answers to ❺ c) and ❺ d).
c) Label your graphs "Galaxy Inn" and "Noble Pines". Describe how the graphs are alike and how they are different.
d) At what value of *n* do the graphs cross? What do you think this means?

❼ Which location, the Galaxy Inn or the Noble Pines Country Club, do you think will offer the lowest total cost for the fundraising party? Explain.

Closure

If you have a graphing calculator viewscreen, display a table of values for Y_1 and Y_2 and show how we can scroll through a table of values to find *n* for which $Y_1 = Y_2$. Then graph these functions on the screen and trace to the point of intersection to verify that $Y_1 = Y_2$ for $n = 30$. Discuss why solving for *n* to make $Y_1 = Y_2$ is the same problem as the two-pan balance problem presented above. Then distribute problem 2 on page 18 and assign it as a homework challenge. As with problem 1, encourage them to involve their parents in the solution.

X	Y1	Y2
30	510	510
31	527	524
32	544	538
33	561	552
34	578	566
35	595	580
36	612	594

X=30

COMPARING THE GALAXY INN WITH THE NOBLE PINES COUNTRY CLUB

DISCUSS

All the exercises in this activity pertain to the quote from the Noble Pines Country Club.

❶ Explain what is meant by "a flat fee of $90." How is this different from a minimum charge?

❷ How much would the Noble Pines Country Club charge for:
a) 10 guests? b) 20 guests? c) 30 guests?

❸ a) How many guests would incur a total cost of $566?
b) What is the largest number of guests that can be accommodated for $1000?
c) How many guests would it take to achieve an average cost of $17 per guest?

All the exercises below pertain to the quote from the Noble Pines Country Club.

❹ a) Make a table showing the cost for each number of guests between 25 and 35.
b) Use your table to check your answer to Exercise ❸ a).
c) Compare your table with the table you constructed in Exercise ❹ of Activity 1. How many guests are needed to make Noble Pines cost per guest less than the Galaxy Inn?

Number of Guests	Cost
25	
26	
•	
•	
•	
34	
35	

❺ a) Describe how you could compute the cost for any given number of guests.
b) Let n represent the number of guests attending the party. Write an expression for the cost for n guests.
c) Use your expression in ❺ b) to calculate the cost for:
(i) 40 guests. (ii) 50 guests. (iii) 55 guests.
d) Use your expression in ❺ b) to calculate how many guests would cost:
(i) \$692. (ii) \$888. (iii) \$1042.

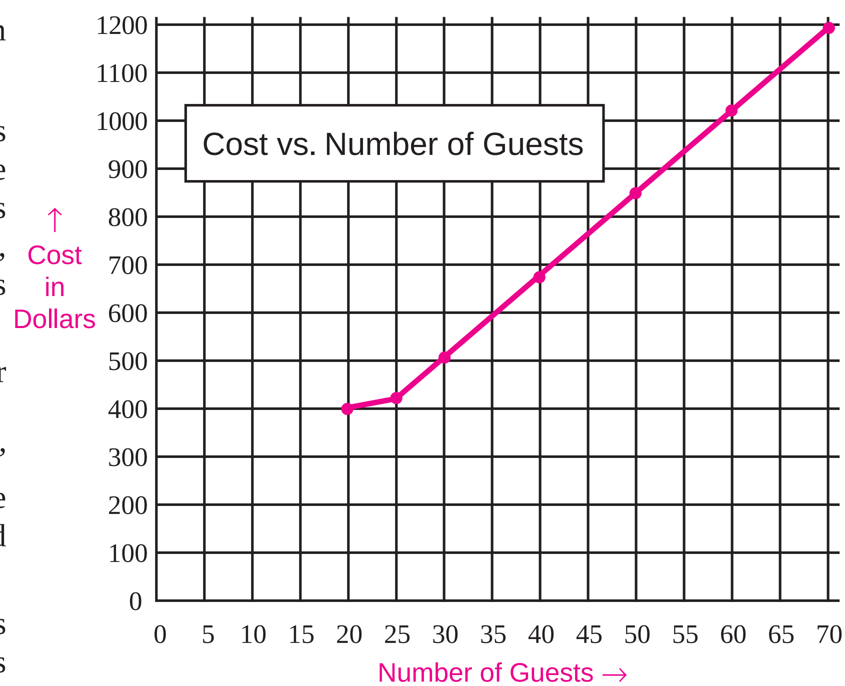
❻ The graph you constructed in Activity 1 should look like this.

a) Using the same grid, plot points in a different colour to show the cost at Noble Pines for n guests for these values of n: 20, 25, 30, 40, 50, 60, 70. Join the new dots in your graph.

b) Use your graph to check your answers to ❺ c) and ❺ d).

c) Label your graphs "Galaxy Inn" and "Noble Pines." Describe how the graphs are alike and how they are different.

d) At what value of n do the graphs cross? What do you think this means?



❼ Which location, the Galaxy Inn or the Noble Pines Country Club, do you think will offer the lowest total cost for the fundraising party? Explain.

Grade 8
Answer Key for Activity 2

❶ A flat fee of \$90 is a fee that is added after the number of guests is multiplied by the cost per guest. This fee is independent of the number of guests. A minimum charge of \$X, on the other hand, specifies that the total cost is the greater of \$X or the number of guests multiplied by the cost per guest. For a sufficiently large number of guests, the cost is merely the product of the number of guests and the cost per guest, and is not affected by the minimum charge.

❷ a) $\$14 \times 10 + \$90 = \$230$
b) $\$14 \times 20 + \$90 = \$370$
c) $\$14 \times 30 + \$90 = \$510$

❸ a) The cost before the flat fee is
$\$566 – \$90 = \$476$
The number of guests is $476 \div 14 = 34$.
34 guests would incur a total cost of \$566.
b) The cost before the flat fee would be:
$\$1000 – \$90 = \$910$
The number of guests would be $910 \div 14 = 65$.
65 guests would incur a total cost of \$1000.
c) Answers will vary. Those students who have not learned how to solve linear equations algebraically are expected to use trial-and-error or to create a table of values of $14n + 90$ (see Exercise 4), beside a table of values of $17n$ and compare. It is important to provide these opportunities for students to explore, before teaching them the formal techniques of algebra. For students who have already learned to solve linear equations, we might expect something like this:

If n guests incur an average cost of \$17, then the total cost of n guests is $\$17n$. But the total cost of n guests at the Noble Pines Country Club is also given by $\$14n + 90$. These are two different ways to write the same total cost, so $17n = 14n + 90$. Solving this equation yields $n = 30$. That is, 30 guests at the Noble Pines Country Club would cost an average of \$17 per guest.

❹ a) The table is shown here.

Number of Guests	Cost
25	440
26	454
27	468
28	482
29	496
30	510
31	524
32	538
33	552
34	566
35	580

b) We observe that the cost of \$566 in the table is opposite 34 guests. This verifies the answer to ❸ a).

c) Comparing the two tables reveals that when there are 30 guests, the costs at the two locations are equal. But when the number of guests is 31 or more, Noble Pines costs less per guest than the Galaxy Inn.

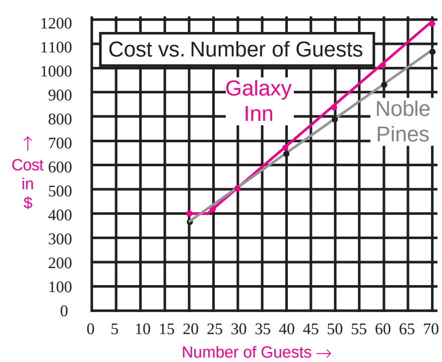
❺ a) Answers will vary, but students should indicate that the number of guests should be multiplied by 14 and this product added to 90.

b) Cost in dollars = $14n + 90$

c) (i) $14 \times 40 + 90 = 650$
(ii) $14 \times 50 + 90 = 790$
(iii) $14 \times 55 + 90 = 860$

d) (i) 43 guests (ii) 57 guests (iii) 68 guests

❻ a) The graph should look something like this.



c) Answers may vary. Students might observe that the graph for the Galaxy Inn is steeper (indicating higher per person cost). Also both graphs are straight lines (for $n \geq 25$).

d) Students will observe that the graphs intersect at $n = 30$. From Exercise ❹, they may know that this means the costs are equal for 30 guests.

❼ Students should realize that Noble Pines offers the lower cost as long as there are more than 30 guests. Some may realize that Noble Pines also offers the lower cost when $n < 23$.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
CONCEPTS **Representation of a Linear Function by:** **• a table** **• a linear equation** **• a graph** (exercises ❹ – ❻) P&A 8-4, 8-8, 8-9, 8-10	•Two or more of the three representations (table of values, equation, or graph) of the Noble Pines C. C. cost function are missing or incorrect.	•One of the three representations (table of values, equation, or graph) of the Noble Pines C. C. is missing or incorrect.	•The correct table of values, equation, and graph are given for the Noble Pines C. C. cost function, with a few minor errors.	•The correct table of values, equation, and graph are given for the Noble Pines C. C. cost function, with almost no errors.
CONCEPTS **Interpretion of the Graph of a Linear Function or Equation** (exercises ❻ c), d) & ❼ P&A 8-6, 8-12	•There is no evidence that the student can interpret the different slopes of the graphs in terms of the costs they imply.	•The student describes the difference in the slope of the two graphs but does not indicate an understanding that each of the locations offers a cheaper rate for different numbers of guests.	•The steeper slope of the Galaxy Inn cost function is interpreted as a higher rate per person. •The student indicates that the Noble Pines C.C. is cheaper when $n > 30$.	In addition to Level 3: •The student interprets correctly the intersection of two lines as the point of equal cost for the same number of guests. •The student recognizes that the Noble Pines is also cheaper when $n < 23$.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

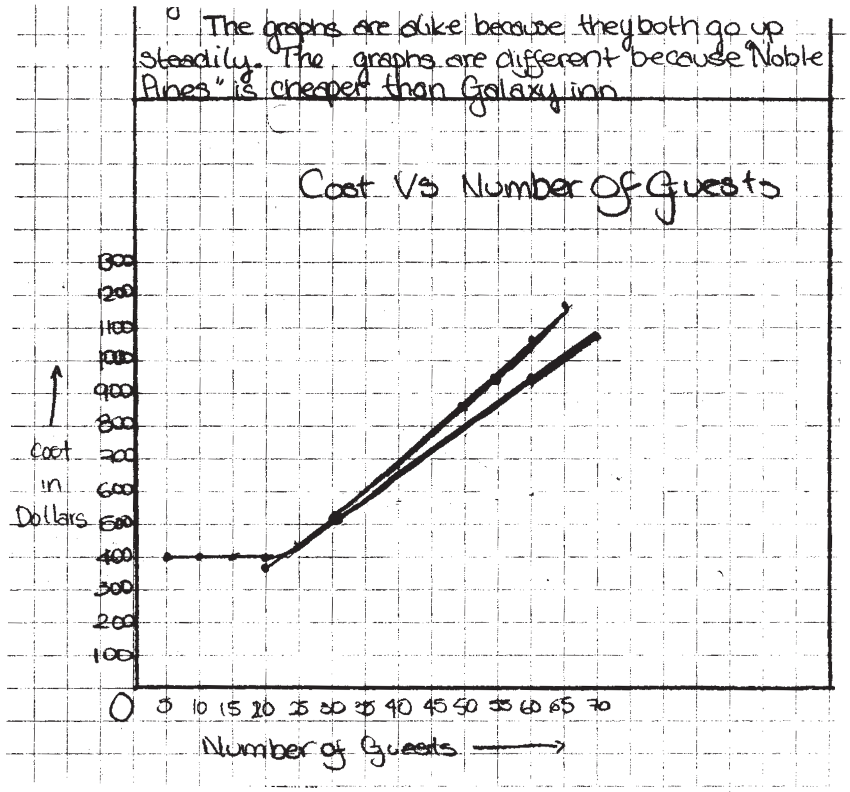
Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

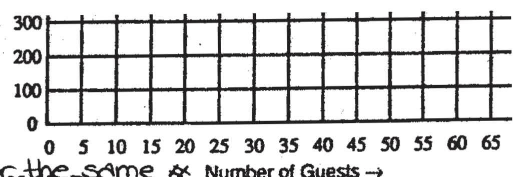
Concepts: Interpretation of the Graph of a Linear Function or Equation

Level 3



c) Describe how the graphs are alike and how they are different. see attached pg

d) At what value of *n* do the graphs cross? What do you think this means? I think it means "Is the cost ever the same for the same # of people?"



❼. Which location, the Galaxy Inn or the Noble Pines Country Club, do you think will offer the lowest total cost for the fundraising party? Explain

I think that Noble Pines will offer the best price because if you look at the graph that we created you could tell that Noble pines was cheaper than Galaxy Inn.

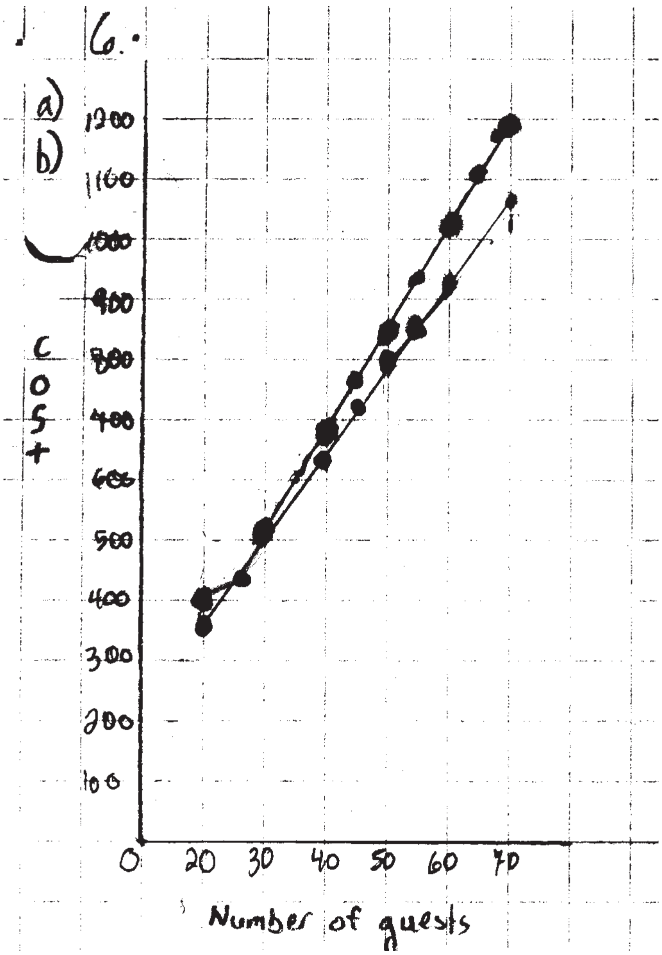
The Galaxy inn would probably be better for 23 people because - 23×14+90=412 - only by 12 dollars but it's still cheaper ....

The graphs of the cost functions for the Galaxy Inn and Noble Pines C. C. are done correctly. The student has observed that both graphs are increasing with *n* ("go up steadily") and that the Noble Pines C.C. offers the lower price. The student also observed that the Galaxy Inn was cheaper for $n = 23$, but stopped short of observing that it is cheaper for $23 \le n < 30$. The student has discovered that the graphs intersect at $n = 30$, and approaches the correct interpretation of an intersection point as one that answers the question, "Is the cost ever the same for the same number of people?"

WHAT YOU MIGHT SEE

Concepts: Interpretation of the Graph of a Linear Function or Equation

Level 4



Activity 2

6c) -after 30 guests both lines rise a fixed rate, but that rate is different for each hotel
-both hotels are the same price for 30 guests
-the Noble Pines always has a lower price, except for 30 guests because that price is the same

6d) the graphs cross at 30 guests, or $510 this means that there is a change in price at some point on the Galaxy Inn graph, because after 30 guests the rates go higher than Noble Pines

Act.2 7. The Noble Pines Country Club offers the best deal for more than 30 guests, and less than 23. That's because of the Galaxy Inn's minimal charge of $400. Anything less than $400 at the Noble Pines is a better deal. BUT. The Galaxy Inn has a better deal for 24-30 guests. I'd still go with the Noble Pines, because there is a bigger range for a better deal

The graphs of the cost functions for the Galaxy Inn and Noble Pines C. C. are done correctly for 20 or more guests. (Students were not asked to extend the graphs to $n < 20$.) In the response to ❻ c), the student has observed that both graphs are "rising at a fixed rate, but that rate is different for each hotel". The student also observes that Noble Pines C.C. is cheaper for $n > 30$. In ❻ d), the student observes that the graphs cross at 30 guests and that this corresponds to a cost of $510 at each location, but expresses it awkwardly as a "change in [average] price". The student has observed that the Galaxy Inn was cheaper for $23 \leq n < 30$, a fact that few students discovered. The student's response to ❼ reveals a deep understanding of the idea that the estimated number of guests is subject to error and that Noble Pines is preferable because it offers the better option over a "bigger range" in the number of guests.

Activity 3 – Teacher Edition

Comparing the Holiday Lodge with the Noble Pines Country Club

Expectations Addressed

- P&A 8-4 use the concept of variable to write equations and algebraic expressions.
- P&A 8-9 translate complex statements into algebraic expressions or equations.
- P&A 8-10 solve and verify first-degree equations with one variable, using various techniques involving whole numbers and decimals.
- P&A 8-12 interpret the solution of a given equation as a specific number value that makes the equation true.

Context

This activity extends the investigation of linear functions in Activity 2 from the exploration of the cost functions $17n$ and $14n + 90$ to the cost function $12(n - 25) + 500$. The lesson development parallels that of Activity 2. After students explain in words how to compute the costs for various numbers of guests at the Holiday Lodge, they are asked in Exercise ❺b) to translate this verbal description into an algebraic expression (see Expectation 8-9). In Exercise ❻, students graph the linear function $12(n - 25) + 500$ on the grid used in Activities 1 and 2. This enables them to compare the graphs of the cost functions of the three sites. In Exercise ❻ d), students are asked to indicate the value of n at which the graphs of the Noble Pines C.C. and the Holiday Lodge cross. They are expected to observe that the graphs cross at $n = 55$, indicating that the costs for both sites are equal for this number of guests. This is tantamount to asking for the graphical solution of the linear equation $14n + 90 = 12(n - 25) + 500$. A formal algebraic solution of this equation is not expected at this point.

The purpose of Activity 3 is to help students create and interpret the tabular and the graphical representations of linear functions in both simplified and unsimplified forms. They will observe that a minimum cost makes that option a more expensive proposition for small values of n, but for a sufficiently large value of n, the per person cost (coefficient of n) is more important. Another objective of this lesson is to help students use inequalities to determine the range in the values of n for which each option is preferred. The cartoon on page 78 is intended to emphasize this point. That is, as the number of guests increases, it is the per person cost (coefficient of n) that has the dominant effect on the total cost, and flat fees and minimum charges become less important.

Activity 3 – Student Page

Comparing The Holiday Lodge With The Noble Pines Country Club

© Taisa Dorney

Discuss

❶. a) Explain what is meant by the statement, "We charge \$500 for the first 25 guests and then \$12 per guest for those in excess of 25"?

b) What would the Holiday Lodge charge if only 23 guests attended the party? What would the charge be for 26 guests?

Activity 3 – Teacher Edition

The Lesson Launch 15 minutes

Before launching this lesson, ask whether anyone was successful in solving the problem handed out at the end of the previous class (problem 2, p. 18). Display a copy of the problem on the overhead projector and discuss what it means when a two-pan scale is in balance. Ask students,

- *Give an algebraic expression for the total mass of the coins on the left pan*; *on the right pan*. [Some students will need help distinguishing between the meanings of $12(n – 25)$ and $12n – 25$.]
- *The two pans are in balance; what does that tell us?*
- *How would you state this algebraically?*
- *Finding the value of n that makes both sides equal is called 'solving for n'; how could we do this?*

Allow the students to struggle with the last question before providing a hint such as, *If we subtract equal quantities from both sides of the equation, are the remaining amounts still equal? How do you know?* Explain to students that they will be solving the problem in a different way as they work through this activity.

To launch the lesson, distribute student pages 78 and 79 and have students read to themselves the cartoon on page 78. Then ask students why the boy's suggestion is not helpful. While his observation is true, it does not tell us how large n must be to make the Holiday Lodge the most economical choice.

Paired Activity 30 minutes

Group the students in the same pairs as for Activity 2. Have each pair complete Exercises ❶ through ❺ with each student recording the answers in his notebook. Ask students, when finished, to complete Exercise ❻ on the grid they used in Activity 2. When almost all students are finished, ask about the value of n at which the graphs for Holiday Lodge and Noble Pines cross. Ask what this means about the costs of the two different sites. Ensure that students understand not only that this is the value of n for which the two sites have the same cost, but also that as the number of guests is increased beyond this, the Holiday Lodge offers the lower cost. If students have access to a graphing calculator, ask them to define $Y_2 = 14X + 90$ and $Y_3 = 12(X – 25) + 500$ and then display a table of values for Y_2 and Y_3. Have them scroll through the table to check their answers to Exercises ❺ c) and ❻ d). Ask them to graph these functions and trace along the graphs to check their answers to Exercise ❻ d).

Activity 3 – Student Page

Comparing The Holiday Lodge With The Noble Pines Country Club

All the exercises in this activity pertain to the quote from the Holiday Lodge.

❷. How much would the Holiday Lodge charge for:
a) 20 guests? b) 30 guests? c) 40 guests?

❸. a) How many guests would incur a total cost of $824?
b) How many guests would it take to reduce the average cost per guest to $17?
c) Is there a number of guests that would result in an average cost per guest of $18? Give reasons for your answer.

❹. a) Make a table showing the cost for each number of guests between 50 and 60.
b) Use your table to check your answer to ❸a).
c) Modify the table you constructed in Exercise ❹ of Activity 2 to show the cost of the Noble Pines Country Club for between 50 and 60 guests. How many guests are needed to make Holiday Lodge cost per guest less than the Noble Pines Country Club?

Number of Guests	Cost
50	
51	
.	
.	
.	
59	
60	

❺. a) Describe how you could compute the cost for any given number of guests.
b) Let n represent the number of guests attending the party. Write an expression for the cost for n guests.
c) Use your expression in ❺ b) to calculate the cost for:
(i) 40 guests. (ii) 50 guests. (iii) 55 guests.
d) Use your expression in ❺ b) to calculate how many guests would cost: (i) $716. (ii) $836. (iii) $944.

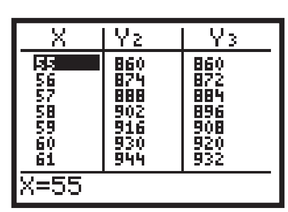
❻. a) On the graph you created in Exercise ❻ of Activity 2, plot points in a different colour showing the cost of n guests for these values of n: 30, 35, 40, 45, 50, 55, 60.
b) Join the dots to make a graph. Label your graph "Holiday Lodge".
c) Describe how all the graphs are alike and how they are different.
d) At what value of n do the Noble Pines C. C. and Holiday Lodge graphs cross? What do you think this means?

➤ *Save your graph for Activity 4.*

❼. Which location, the Noble Pines Country Club or the Holiday Lodge, do you think will offer the lowest total cost for the fundraising party? Explain.

Closure

If you have a graphing calculator viewscreen, display a table of values for Y_2 and Y_3 and show how we can scroll through a table of values to find n for which $Y_2 = Y_3$. Then graph these functions on the screen and trace to the point of intersection to verify that $Y_2 = Y_3$ for $n = 55$. Discuss why solving for n to make $Y_2 = Y_3$ is the same problem as the two-pan balance problem 2 presented above. Distribute copies of p. 93 and discuss the solutions.



Comparing the Holiday Lodge with the Noble Pines Country Club

Discuss

❶ a) Explain what is meant by the statement, "We charge $500 for the first 25 guests and then $12 per guest for those in excess of 25"?

b) What would the Holiday Lodge charge if only 23 guests attended the party? What would the charge be for 26 guests?

Comparing the Holiday Lodge with the Noble Pines Country Club

All the exercises below pertain to the quote from the Holiday Lodge.

❷ How much would the Holiday Lodge charge for:
a) 20 guests? b) 30 guests? c) 40 guests?

❸ a) How many guests would incur a total cost of $824?
b) How many guests would it take to reduce the average cost per guest to $17?
c) Is there a number of guests that would result in an average cost per guest of $18? Give reasons for your answer.

❹ a) Make a table showing the cost for each number of guests between 50 and 60.
b) Use your table to check your answer to Exercise ❸ a).
c) Modify the table you constructed in Exercise ❹ of Activity 2 to show the cost of the Noble Pines Country Club for between 50 and 60 guests. How many guests are needed to make Holiday Lodge cost per guest less than the Noble Pines Country Club?

Number of Guests	Cost
50	
51	
•	
•	
•	
59	
60	

❺ a) Describe how you could compute the cost for any given number of guests.
b) Let n represent the number of guests attending the party. Write an expression for the cost for n guests.
c) Use your expression in ❺ b) to calculate the cost for:
(i) 40 guests. (ii) 50 guests. (iii) 55 guests.
d) Use your expression in ❺ b) to calculate how many guests would cost:
(i) $716. (ii) $836. (iii) $944.

❻ a) On the graph you created in Exercise ❻ of Activity 2, plot points in a different colour showing the cost of n guests for these values of n : 30, 35, 40, 45, 50, 55, 60.
b) Join the dots to form a graph. Label your graph "Holiday Lodge."
c) Describe how all the graphs are alike and how they are different.
d) At what value of n do the Noble Pines C. C. and Holiday Lodge graphs cross? What do you think this means?

➤*Save your graph for Activity 4.*

❼ Which location, the Noble Pines Country Club or the Holiday Lodge, do you think will offer the lowest total cost for the fundraising party? Explain.

Grade 8

Answer Key for Activity 3

❶ a) There is a minimum charge of \$500, and after that charge, the cost for each guest from the 26th on is \$12. In mathematical terms, if $n \le 25$, the cost is \$500. When $n > 25$ the cost is \$12 for each additional guest beyond the first 25 plus the initial \$500.

b) 23 guests would cost \$500, and 26 guests would cost \$500 + \$12 = \$512.

❷ a) \$500
b) \$500 + \$12 × 5 = \$560
c) \$500 + \$12 × 15 = \$680

❸ a) The cost for the guests beyond the first 25 is \$824 – \$500 = \$324. This represents 324 ÷ 12 or 27 guests beyond the first 25. So \$824 would be the cost of 25 + 27 or 52 guests.
b) Answers will vary. Students who have not learned how to solve linear equations algebraically are expected to use trial-and-error or to create a table of values of $500 + 12(n - 25)$ (see Exercise ❹), beside a table of values of $17n$ and compare. It is important to provide these opportunities for students to explore before teaching them the formal techniques of algebra. An exceptional student might graph both these functions and look for a point of intersection. For those who have already learned to solve linear equations, we might expect something like this:

If n guests incur an average cost of \$17, then the total cost of n guests is \$17n. But the total cost of n guests at the Holiday Lodge is also given by \$500 +\$12(n – 25). These are two different ways to write the same total cost, so:

$$17n = 500 + 12(n - 25).$$

Solving this equation yields $n = 40$. That is, 40 guests at the Noble Pines Country Club would cost an average of \$17 per guest.
c) Any of the methods described in Part b) could be used to determine that there is no integral value for n for which the expressions $500 + 12(n - 25)$ and $18n$ assume the same value. Depending on the method used, students may indicate that the average cost per guest at the Holiday Lodge is greater than \$18 for $n \le 33$ and less than \$18 for $n > 33$.

❹ a) This is the table.

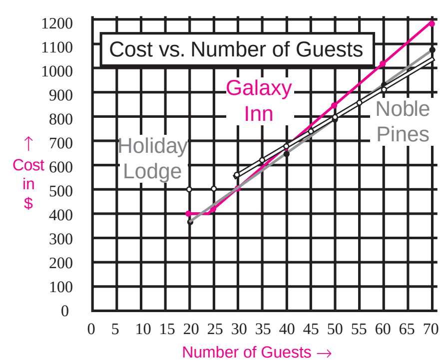
Number of Guests	Cost at Holiday Lodge	Cost at Noble Pines
50	800	790
51	812	804
52	824	818
53	836	832
54	848	846
55	860	860
56	872	874
57	884	888
58	896	902
59	908	916
60	920	930

b) We observe that the cost of \$824 in the table is opposite 52 guests. This verifies the answer to ❸ a).

c) Comparing the two columns reveals that the costs are equal for 55 guests. But when the number of guests is 56 or more, the Holiday Lodge cost per guest is less than that of Noble Pines.

❺ a) Answers will vary, but students should indicate that 25 is to be subtracted from the number of guests, and the difference should be multiplied by 12. Then 500 should be added to the result.
b) Cost in dollars = $12(n - 25) + 500$ for $n > 25$ and \$500 for $n \le 25$.
c) (i) \$12 × (40 – 25) + \$500 = \$680
(ii) \$12 × (50 – 25) + \$500 = \$800
(iii) \$12 × (55 – 25) + \$500 = \$860
d) (i) 43 guests (ii) 53 guests (iii) 62 guests

❻ a) & b) The graph should look something like this.



c) Answers may vary. Students might observe a difference in steepness (slope). Also all graphs are straight lines for sufficiently large n.
d) Students will observe that the graphs intersect at $n = 55$. From Exercise ❹, they may know that this means the costs are equal for 55 guests.

❼ Students should recognize from the graph or the table that the Holiday Lodge offers the lower cost for more than 55 guests.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

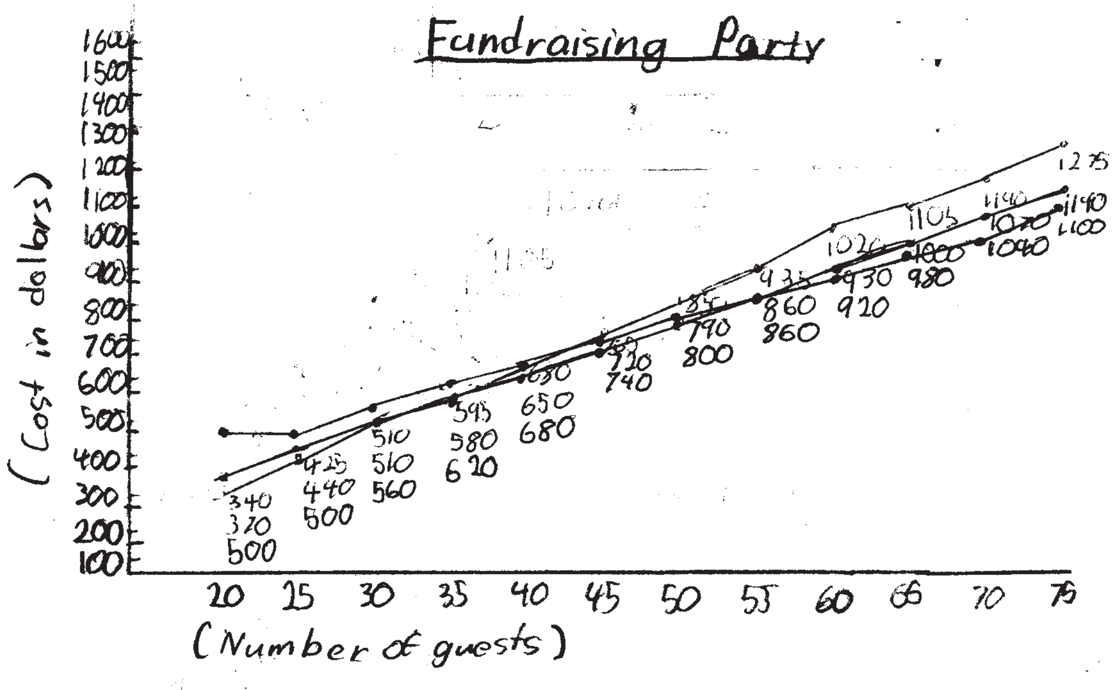
Scoring Guide for Activity 3

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Solution of Problems by Applying an Appropriate Strategy** (exercises ❸, ❹, ❺, ❼) P&A 8-7	•Applies a limited range of strategies to the solution of problems and frequently makes errors.	•Usually solves problems using an appropriate strategy but with several minor errors.	•Solves problems using an appropriate strategy almost always and with few minor errors.	•Always solves problems using an appropriate strategy and sometimes employs a creative new strategy.
CONCEPTS **Representation of a Linear Function by:** **• a table** **• a linear equation** **• a graph** (exercises ❹ – ❻) P&A 8-4, 8-8, 8-9, 8-10	•Two or more of the three representations (table of values, equation, or graph) of the Holiday Lodge cost function are missing or incorrect.	•One of the three representations (table of values, equation, or graph) of the Holiday Lodge is missing or incorrect.	•The correct table of values, equation, and graph are given for the Holiday Lodge cost function, with a few minor errors.	•The correct table of values, equation, and graph are given for the Holiday Lodge cost function, with almost no errors.
APPLICATION **Solution of a Linear Equation** **• Algebraically** **• Graphically** (exercises ❺ d) & ❻ b) P&A 8-10	•There is no evidence of a correctly constructed linear equation corresponding to the cost of a particular number of guests.	•The appropriate equation is constructed, but there is evidence of difficulty in solving it to obtain the number of guests corresponding to a given cost.	•Determines algebraically the number of guests corresponding to given costs at the Holiday Lodge and verifies graphically the algebraic solution of a linear equation, with a few minor errors.	•And in addition to Level 3, interprets correctly the intersection of two lines as the point of equal cost for the same number of guests.

WHAT YOU MIGHT SEE

Problem Solving: Solution of Problems by Applying an Appropriate Strategy

Level 1



This student drew the graphs correctly and shows some competence in solving one-step problems. For example, the student answered Exercises ❶, ❷, ❸ a), and ❺ c) correctly, using the tables of values. However, the student seems unable to apply an appropriate strategy to the solution of more difficult problems, such as ❸ b) and c). To determine how many people it would take to average $17 per person at Holiday Lodge, the student divided $500 by 17 to obtain the estimate 27.7. Also, in Exercise ❹ c), the student was unable to determine, even from the tables of values, the number of guests that would make the Holiday Lodge the less expensive option. In Exercise ❼, the student correctly chooses the Holiday Lodge as the cheaper option, but does so by comparing costs for 60 guests–a single one-point comparison.

❸. a) How many guests would incur a total cost of $824? 52 guests
b) How many guests would it take to reduce the average cost per guest to $17? 29 guests
c) Is there a number of guests that would result in an average cost per guest of $18? Give reasons for your answer. 27 guests, because 500 ÷ 17 = 27.7

❹. a) Make a table showing the cost for each number of guests between 50 and 60. ✓
b) Use your table to check your answer to Exercise ❸. a). ✓
c) Modify the table you constructed in Exercise ❹ of Activity 2 to show the cost of the Noble Pines Country Club for between 50 and 60 guests. How many guests are needed to make Holiday Lodge cost per guest less than the Noble Pines Country Club? 46 guests

Number of Guests	Cost
50	[illegible]
51	[illegible]
•	[illegible]
•	[illegible]
•	[illegible]
59	[illegible]
60	[illegible]

❺. a) Describe how you could compute the cost for any given number of guests.
b) Let *n* represent the number of guests attending the party. Write an expression for the cost for *n* guests. n ÷ 25 × 12 + 500 =
c) Use your expression in Part b) to calculate the cost for:
(i) 40 guests $650 (ii) 50 guests $800 (iii) 55 guests $860

❻. a) On the graph you created in Exercise ❻ of Activity 2, plot points in a different colour showing the cost of *n* guests for these values of *n* : 30, 35, 40, 45, 50, 55, 60. ✓
b) Join the dots to form a graph. Label your graph "Holiday Lodge".
c) Describe how all the graphs are alike and how they are different.
d) At what value of *n* do the Noble Pines C. C. and Holiday Lodge graphs cross? What do you think this means? $510 I am not quite sure, but I know it means that everyone in excess of 30 guests are getting a better deal at Novel Pines

❼. Which location, the Noble Pines Country Club or the Holiday Lodge, do you think will offer the lowest total cost for the fundraising party? Explain.
I think the holiday lodge would be better, because
HL = 60g's = 920
NP = 60g's = 1020

7

WHAT YOU MIGHT SEE

Problem Solving: Solution of Problems by Applying an Appropriate Strategy

Level 4

Not only did this student answer all of the exercises correctly, but she shows some flamboyance in her answer to Exercise ❸ c). She determines that 33 guests yield an average cost of $17.88 and 34 guests yield an average cost of $18.06, and then asserts ("Since you can't divide a guest into pieces…") it is not possible to attain an average cost per guest of exactly $18.00.

In Exercise ❺, the student not only describes correctly the algorithm for calculating the cost function, but she gives a correct algebraic expression to define it, and applies it correctly in her calculations.

Finally, in Exercise ❼, she is able to describe correctly the range of values of *n* for which each location is the less expensive option. In general, this student's responses indicate that she understands both the algebraic and graphical representations of the cost functions and is able to apply them in the solution of problems.

3.c. No, not exactly, because 33 guests makes the average cost $18.06 and 34 guests makes the average cost $17.88. Since you can't divide a guest into pieces, you will never get exactly $18.00 for an average cost.

4.a.

Number of Guests	Cost (in dollars)
50	$800
51	$812
52	$824
53	$836
54	$848
55	$860
56	$872
57	$884
58	$896
59	$908
60	$920

HOLIDAY LODGE

Number of Guests	Cost (in dollars)
50	$790
51	$804
52	$818
53	$832
54	$846
55	$860
56	$874
57	$888
58	$902
59	$916
60	$930

NOBLE PINES

5.a. If the number of guests is less or exactly 25, the charge is $500.00. If the number of guests is 26 or more, you subtract 25 and then multiply by 12. Finally, you add 500. The answer is the cost, in dollars.

❺. a) Describe how you could compute the cost for any given number of guests. (on lined paper) Pg 4
b) Let *n* represent the number of guests attending the party. Write an expression for the cost for *n* guests. (when $n > 25$) → $(n-25)12 + 500$
c) Use your expression in Part b) to calculate the cost for:
(i) 40 guests. (ii) 50 guests. (iii) 55 guests.
$680.00 $800.00 $860.00

❻. a) On the graph you created in Exercise ❻ of Activity 2, plot points in a different colour showing the cost of *n* guests for these values of *n* : 30, 35, 40, 45, 50, 55, 60.
b) Join the dots to form a graph. Label your graph "Holiday Lodge". (on blank paper.) Pg. 5
c) Describe how all the graphs are alike and how they are different.
d) At what value of *n* do the Noble Pines C. C. and Holiday Lodge graphs cross? What do you think this means? When $n = 60$. I think this means that after this point, Holiday Lodge runs lower than Noble Pines, which means that when $n > 60$, Holiday Lodge is cheaper.

❼. Which location, the Noble Pines Country Club or the Holiday Lodge, do you think will offer the lowest total cost for the fundraising party? Explain.
I think Holiday Lodge, because the price is the cheapest for 70 people. Noble Pines is the best location, if there are less than 55 guests, but if there are more, (as in Jennifer's case), Holiday Lodge is the best Choice, in terms of looking at price.

79

Activity 4 – Teacher Edition

Making a Choice

Expectations Addressed

- P&A 8-4 use the concept of variable to write equations and algebraic expressions.
- P&A 8-5 investigate inequalities and test whether they are true or false by substituting whole number values for the variables (e.g., in $4x \geq 18$, find the whole number values for x).
- P&A 8-6 write statements to interpret simple equations.
- P&A 8-10 solve and verify first-degree equations with one variable, using various techniques involving whole numbers and decimals.
- P&A 8-12 interpret the solution of a given equation as a specific number value that makes the equation true.
- DM 8-9 discuss the quantitative information presented on tally charts, frequency tables, and/or graphs.
- DM 8-15 construct line graphs, comparative bar graphs, circle graphs, and histograms, with and without the help of technology, and use the information to solve problems.

Context

The purpose of this activity is to help students use inequalities to define the intervals in which each of the quotes is the most economical. This will enable them to resolve the intial question, *Where should they hold the fundraising party?*

The student reports should indicate that Jennifer estimated that 20 to 25% of the 80 invited guests would not attend. This means that between 60 and 64 guests would attend. In their response to Exercise ❷ c), the students should indicate that Holiday Lodge offers the lowest total cost when there are more than 55 guests. Consequently the report should indicate that the Holiday Lodge is likely to be the most economical choice. However, it is important that students indicate in their report that this choice is based on Jennifer and Steve's estimate of the number of guests, and that a substantially smaller attendance would render the Holiday Lodge a less than optimal choice.

The spreadsheet activity on page 87 has a dual purpose. The first is to show students how a kind of algebraic notation is used to create a spreadsheet to compare the three cost functions. The second purpose is to acquaint the students with how spreadsheets can be used to conduct a *What if?* scenario. In this activity, students discover that if the per person cost is reduced by $2 in each quote, the Noble Pines Country Club becomes the most economical choice.

Activity 4 – Student Page

Making a Choice

❶ Get the graph you created in Activity 3 showing the cost for any number of guests between 20 and 70 at each of the three locations.
a) Write the equation that defines each line.
b) Write the coordinates of each point where two lines intersect.
c) Explain the significance of each point in ❶ b).

❷ The Galaxy Inn offers the lowest total cost of all three locations when the number of guests (denoted by n) is greater than 22 and less than 30. To state this, we write:
The Galaxy Inn offers the lowest total cost for $22 < n < 30$.

Write an inequality to show the values of n for which:
a) the Noble Pines Country Club offers the lowest cost.
b) the Holiday Lodge offers the lowest cost.

❸ Create tables to verify your answers to Exercise ❷.

❹

Research Report

Based on attendance at past events, Jennifer and Steve's committee believes that between 20 and 25% of the 80 people invited will not attend. As secretary of the Fundraising Committee, write a report to the faculty advisor, Ms. Vohra, recommending one of the three locations.

Support your recommendation by including the following elements in your report:

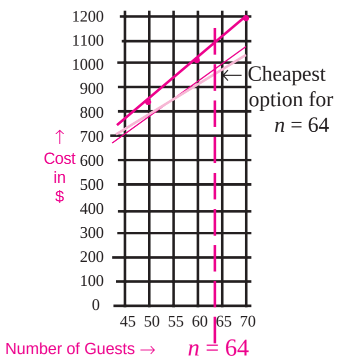
- the approximate number of guests that are expected to attend.
- the algebraic expressions that give the costs at each location for n guests.
- the cost at each location for the estimated number of guests
- graphs and/or tables to show the range in the number of guests for which your recommendation offers the lowest cost.
- possible reasons why the location offering the lowest cost might not bethe best choice.

ACTIVITY 4 – TEACHER EDITION

The Lesson Launch
5 minutes

To launch this lesson, ask students questions such as:

- *State an algebraic expression that gives the cost function for n guests at:*
  - *the Galaxy Inn;*
  - *the Noble Pines Country Club;*
  - *the Holiday Lodge.*
- *Which of these locations is cheapest if only 10 guests attend? Explain.*
- *How can you tell from the graphs of the cost functions which location is the least expensive for a given number of guests?*



Through discussion, students should come to understand how they can determine the least expensive option for n_0 guests by drawing a vertical line at $n = n_0$ and identifying the cost function intersecting this line at the lowest point. They should also see how inequalities can be used to show the range of values of n for which each choice is optimal. These skills are an essential component of *graph literacy.*

Paired Activity
35 minutes

Using the same pair groupings as for Activities 2 and 3, have students collaborate in the completion of Exercises ❶ through ❸ on page 86. Check that students are able to identify the points of intersection of the graphs and describe the values of n for which each location is the best choice. Ask students, when finished, to work together on their research report.

Hand out page 87 to those students who finish their reports first, and ask them to proceed with the spreadsheet activity.

Closure

When all groups have finished preparing their reports, have several students (preferably with different answers) share their choice of location and provide reasons. Encourage debate about what factors other than cost might affect the choice of location. Discuss also how an error in the estimate of the number of guests might affect the choice of location. Ask students why a hotel or lodge imposes a minimum charge or a flat fee. Challenge them to create a quote that is less expensive than all three of the given quotes for $n > 60$, but more expensive for $n \leq 60$.

Assign the spreadsheet activity (p. 87) as a project for each group. Attach a due date about 14 days hence. This should provide enough time for each pair to gain access to the computer and to print out their results.

ACTIVITY 4 – STUDENT PAGE

USING A SPREADSHEET TO ASK WHAT IF? …

Now that you can express relationships using algebra, you can use a spreadsheet to create at table of values. Follow these steps to create a table of values to compare the costs of the original quotes for various numbers of guests.

❶ Use a program such as *Claris Works* to create a spreadsheet with headings like those shown here.

	A	B	C	D
1	The Costs of the Three Locations			
2	vs. the Number of Guests			
3	Guests	Galaxy Inn	Noble Pines	Holiday Lodge

❷ To prepare to enter formulas for the cost at each location, choose **Display…** from the **Options** menu and then select **Formulas** from that dialogue box. To start the table at 25 guests, enter =25 in cell A4. Then enter the cost formulas in row 4 as shown.

	A	B	C	D
1	The Costs of the Three Locations			
2	vs. the Number of Guests			
3	Guests	Galaxy Inn	Noble Pines	Holiday Lodge
4	=25	=17*A4	=90+14*A4	500+12(A4-25)

❸ To extend the formulas from 25 to 60, select cell A5 and enter the formula **=A4 + 1**. Highlight that column up to the cell A60. Then select **Fill Down** from the **Calculate menu**. Then repeat this process for rows B, C and D. Rows 3, 4, 5, and 6 are shown.

3	Guests	Galaxy Inn	Noble Pines	Holiday Lodge
4	=25	=17*A4	=90+14*A4	500+12(A4-25)
5	=A4+1	=17*A5	=90+14*A5	500+12(A5-25)
6	=A5+1	=17*A6	=90+14*A6	500+12(A6-25)

To replace the formulas with numbers, deselect the option **Formulas** from the **Display** command on the **Options** menu. You can now compare costs.

WHAT IF?

Before making your final decision, you decide to call all three locations and offer to pay two dollars less per guest than they quoted. That is, you offer to pay Galaxy Inn \$15 per guest with a minimum total of \$400, Noble Pines \$90 plus \$12 per guest and Holiday Lodge \$500 plus \$10 per guest for the number of guests in excess of 25. All three locations agree to the lower rate. Does this change your decision? Explain your answer.

MAKING A CHOICE

❶ Get the graph you created in Activity 3 showing the cost for various numbers of guests at each of the three locations.
a) Write the equation that defines each line.
b) Write the coordinates of each point where two lines intersect.
c) Explain the significance of each point in ❶ b).

❷ The Galaxy Inn offers the lowest total cost of all three locations when the number of guests (denoted by *n*) is greater than 22 and less than 30. To state this, we write:

The Galaxy Inn offers the lowest total cost for $22 < n < 30$.

Write an inequality to show the values of *n* for which:
a) the Noble Pines Country Club offers the lowest cost.
b) the Holiday Lodge offers the lowest cost.

❸ Create tables to verify your answers to Exercise ❷.

❹ **RESEARCH REPORT**

Based on attendance at past events, Jennifer and Steve's committee believes that between 20 and 25% of the 80 people invited will not attend. As secretary of the Fundraising Committee, write a report to the faculty advisor, Ms. Vohra, recommending one of the three locations.

Support your recommendation by including the following elements in your report:
- the approximate number of guests who are expected to attend.
- the algebraic expressions that give the costs at each location for *n* guests.
- the cost at each location for the estimated number of guests
- graphs and/or tables to show the range in the number of guests for which your recommendation offers the lowest cost.
- possible reasons why the location offering the lowest cost might not be the best choice.

Using a Spreadsheet to Ask What if? ...

Now that you can express relationships using algebra, you can use a spreadsheet to create at table of values. Follow these steps to create a table of values to compare the costs of the original quotes for various numbers of guests.

1 Use a program such as *Claris Works* to create a spreadsheet with headings like those shown here.

	A	B	C	D
1		The Costs of the Three Locations		
2		in Terms of the Number of Guests		
3	Guests	Galaxy Inn	Noble Pines C.C.	Holiday Lodge

2 To prepare to enter formulas for the cost at each location, choose **Display...** from the **Options** menu and then select **Formulas** from that dialogue box. To start the table at 25 guests, enter =25 in cell A4. Then enter the cost formulas in row 4 as shown.

	A	B	C	D
1		The Costs of the Three Locations		
2		in Terms of the Number of Guests		
3	Guests	Galaxy Inn	Noble Pines C.C.	Holiday Lodge
4	=25	=17*A4	=90+14*A4	=500+12*(A4-25)

3 To extend the formulas from 25 to 60, select cell A5 and enter the formula **=A4 + 1**. Highlight that column up to the cell A60. Then select **Fill Down** from the **Calculate menu**. Then repeat this process for rows B, C and D. Rows 3, 4, 5, and 6 are shown.

3	Guests	Galaxy Inn	Noble Pines C.C.	Holiday Lodge
4	=25	=17*A4	=90+14*A4	=500+12*(A4-25)
5	=A4+1	=17*A5	=90+14*A5	=500+12*(A5-25)
6	=A5+1	=17*A6	=90+14*A6	=500+12*(A6-25)

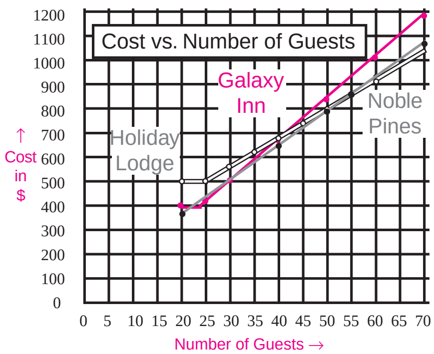
To replace the formulas with numbers, deselect the option **Formulas** from the **Display** command on the **Options** menu. You can now compare costs.

What If?

Before making your final decision, you decide to call all three locations and offer to pay two dollars less per guest than they quoted. That is, you offer to pay Galaxy Inn $15 per guest with a minimum total of $400, Noble Pines $90 plus $12 per guest and Holiday Lodge $500 plus $10 per guest for the number of guests in excess of 25. All three locations agree to the lower rate. Does this change your decision? Explain your answer.

GRADE 8
ANSWER KEY FOR ACTIVITY 4

❶ The graph should look something like this.



a) The equations that define each line (not including any horizontal line segments) are:

Galaxy Inn Y = 17n $n > 23$

Noble Pines Country Club Y = 14n + 90

Holiday Lodge Y = 500 + 12(n – 25) $n > 25$

b) The Noble Pines and Galaxy Inn lines intersect at the point (30, 510).
The Holiday Lodge and Galaxy Inn lines intersect at the point (40, 680).
The Holiday Lodge and the Noble Pines lines intersect at the point (55, 860).

c) The intersection point (n, m) gives the number of guests n at which both cost functions take the same value, m dollars.

❷ a) The Noble Pines offers the lowest total cost when the number of guests is fewer than 23, and when the number of guests is between 31 and 54. That is, the Noble Pines offers the lowest cost for $n < 23$ or for $31 \le n \le 54$.

b) The Holiday Lodge offers the lowest total cost when the number of guests is greater than 55, i.e., $n > 55$.

❸ This table shows the costs at each location for up to 30 guests. By checking this table, we can verify the answer to Exercise ❷ a) and that Noble Pines offers the lowest cost for $n < 23$. The tables on pages 72 and 80 verify that Noble Pines offers the lowest cost for $31 \le n \le 54$ and the Holiday Lodge offers the lowest price for $n > 55$.

Number of Guests	Cost at Galaxy Inn	Cost at Noble Pines	Cost at Holiday Lodge
1	400	104	500
2	400	118	500
3	400	132	500
4	400	146	500
5	400	160	500
6	400	174	500
7	400	188	500
8	400	202	500
9	400	216	500
10	400	230	500
11	400	244	500
12	400	258	500
13	400	272	500
14	400	286	500
15	400	300	500
16	400	314	500
17	400	328	500
18	400	342	500
19	400	356	500
20	400	370	500
21	400	384	500
22	400	398	500
23	400	412	500
24	408	426	500
25	425	440	500
26	442	454	512
27	459	468	524
28	476	482	536
29	493	496	548
30	510	510	560

❹ The student report should include all the elements listed in the specifications. It is expected that graphs, tables or inequalities will be incorporated in the report to justify the site selected.

Some creativity should be shown in suggesting why the least expensive site may not be the best choice; e.g. limited facilities, poor location, uncooperative staff.

USING A SPREADSHEET TO ASK WHAT IF?

Under the new arrangements with each per person rate decreased by $2, the Noble Pines C. C. is the least expensive for $n \le 25$ and for $31 \le n \le 79$. The Galaxy Inn is the least expensive for $26 \le n \le 29$, and the same as Noble Pines for $n = 30$. The Holiday Lodge is least expensive for $n > 80$ and the same as Noble Pines for $n = 80$. It would seem that the Noble Pines Country Club is likely to offer the least cost for almost any number of guests.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 4

	Level 1	Level 2	Level 3	Level 4
CONCEPTS				
Use of Inequalities to Describe a Range of Values of a Variable that Satisfy a Condition (exercises ❷ – ❸) P&A 8-5, 8-12	•The student has significant difficulty identifying the correct range of values of *n* (the number of guests) for which each of the three locations offers the least cost.	•The student describes, with several errors, the range of values of *n* (the number of guests) for which each of the three locations offers the least cost.	•The student describes, with almost no errors, the correct range of values of *n* (the number of guests) for which each of the three locations offers the least cost.	•In addition to Level 3, the student uses formal inequalities of the form $a < n < b$ to describe each range of values of *n*.
COMMUNICATION				
Report with Quantitative Support of a Recommendation Involving Comparative Quotes (exercise ❹) P&A 8-7, 8-12	•The report is poorly organized and/or unclear. •One or more of the elements delineated in Level 3 is missing.	•The report is missing one of the elements delineated in Level 3.	•The report presents clearly, correctly, and coherently these elements: – assumed number of guests. – number of guests for which each location is least expensive. – recommended location.	• And in addition to Level 3, the report considers alternatives, limitations of relying entirely on the quotes, and a possible error in the estimated number of guests.

WHAT YOU MIGHT SEE

Report with Quantitative Support of a Recommendation involving Comparative Quotes

Level 1

This report presents the correct algebraic expression for each of the three cost functions. The table of values is correct except for a consistent error of $10 in the costs for the Noble Pines Country Club. There is no reasonable estimate of the number of guests who will attend, and no ranges given for *n* that define the number of guests for which each location is cheapest. The Noble Pines Country Club is recommended, but it is not supported with anything other than the observation that more than 20 guests will probably come. This conflicts with the suggestion that the Galaxy Inn has a better price for "less than 30 people." The explanation is unclear and incoherent.

Dear Ms. Vohra,

I think that the Noble Pines Country Club offers the best deal because more than 20 people will probably come. I think the Noble Pines Country Club is safer because it would be difficult to cover for the total cost for the Holiday Lodge. You would have to find the exact number of people coming before you can find a fair price for each person. There would have to be less than 30 people for the Galaxy Inn to have a better price. The Galaxy Inn offers a price of *nx17* but the minimum charge is $400. The Noble Pines offers a price of *nx14+90* and the Holiday Lodge offers a price of *(n-25)x12+500*.

Here is a table to show the prices from 55 to 65 guests.

Number of Guests	Noble Pines	Holiday Lodge	Galaxy Inn
55	$850	$860	$935
56	$864	$872	$952
57	$878	$884	$969
58	$892	$896	$986
59	$906	$908	$1003
60	$920	$920	$1020
61	$934	$932	$1037
62	$948	$944	$1054
63	$962	$956	$1071
64	$976	$968	$1088
65	$990	$980	$1105

Level 3

This report contains the three elements presented in the Level 3 criteria of the scoring guide on p. 89. The number of guests is estimated to be between 60 and 64. The ranges of cost for these numbers of guests are correctly quoted for each location, and the Holiday Lodge is given as the least expensive location for *n* > 55. The report also cautions that the estimate may not be accurate and that if fewer than 55 guests attend, then the Holiday Lodge would not be the least expensive alternative. The ranges of *n* for which each of the other locations is least expensive was not given and there was no discussion of why cost might not be the only consideration in choosing a location.

Ms. Vohra,

I have researched this decision long and hard. I think I have come up with a good place to have our fundraising party. Based on past events there will be about 60-64 that will come out of the 80 invited. The holiday lodge offers the best price for 60-64 people.

Galaxy Inn - $1020 - $1088
Nobel Pines - $930 - $986
★ Holiday Lodge - $920 - $968 ★

The only possible problem with the Holiday Lodge, is if less than 55 people come, you will have to pay more because the price is higher than the others at 55-.

WHAT YOU MIGHT SEE

Report with Quantitative Support of a Recommendation involving Comparative Quotes

Level 4

This report contains the three elements presented in the Level 3 criteria of the scoring guide on p. 89. The number of guests is estimated to be between 60 and 64. The algebraic expressions that define the cost functions are also presented correctly. The ranges of cost for between 60 and 64 guests are correctly quoted for each location, and the Holiday Lodge is recommended as the least expensive location for $n > 55$. The student has also indicated that the Noble Pines C.C. is the better choice for $n < 55$. The student also correctly indicates that the Galaxy Inn is cheapest for $n = 23$. (Actually it is cheapest for $23 \le n < 30$.) The report also cautions that the estimate may not be accurate and that if fewer than 55 guests attend, then the Holiday Lodge would not be the least expensive alternative. There was no discussion of why cost might not be the only consideration in choosing a location. While the scoring guide suggests that such a discussion might be a requisite for a Level 4 assessment, we waive this requirement on this report because the student's understanding of the inequalities defining the optimal locations for various numbers of guests was so well articulated.

The decision above to bend the strict requirements of the scoring guide to account for exceptional or unexpected student responses reflects the "organic" nature of scoring guides. As assessment instruments, they are merely guides rather than straightjackets and they must evolve and change as they are used.

Dear Ms. Vohra,

The approximate number of guests that are expected to attend are 60 to 64. Here are three algebraic expressions that give the costs at each location when "n" represents the number of guests.

Galaxy Inn: (where n ≤ 23) $400, (where n > 23) 17n

Noble Pines C.C.: 14n + 90

Holiday Lodge: (where n ≤ 25) $ 500.00

(where n > 25) → (n − 25)12 + 500

I have calculated the cost at each location for the estimated number of guests. At the Galaxy Inn, you will pay $1,020.00 to 1088.00. At the Noble Pines C.C., you will pay $930.00 to $986.00. At the Holiday Lodge, you will pay $920.00 to $ 968.00. I believe that the Holiday Lodge is the best location to hold the fundraising party if there ar over 55 guests. But if there are less, the Noble Pines C.C. is a better place. Look:

COST

# of Guests	Galaxy Inn	Noble Pines	Holiday L.
64	$1,088	$986	$ 968
60	$1,020	$930	$920
53	$ 901	$832	$836
54	$918	$846	$848

If the number of guests is more than 55 guests, Holiday Lodge. If there are 54 or less guests: Noble Pines. If there 23 guests (not on this table, Galaxy Inn.)

The reason for which Holiday Lodge might not be the best choice is if attendance falls below 55 guests. Because we are only predicting the number of guests to show up, we are not 100 percent certain, ~~if~~ and this is likely, attendance drops below 55, the Noble Pines C.C. is a better choice.

Nevertheless I would still go with the Holiday Lodge because we are inviting 80 guests. Let's hope that everybody will come.

Yours truly,

Jennifer

secretary of Fund-Raising Comittee

WHAT YOU MIGHT SEE

Report with Quantitative Support of a Recommendation involving Comparative Quotes

Level 4

Friday, October 2, 1998

Dear Ms. Vohra,

Hi, I am writing to you about the fund raising party. I have checked the prices of three hotels, the Holiday Lodge, Noble Pines Country Club and the Galaxy Inn and I have found out the prices.

Based on past attendance records, I have found that only about 20 to 25 percent of the people invited will not come. This means, that of the 80 students invited, only 60 to 64 will attend the party. I used the average number 62 in my calculations to find the cheapest deal and I came up with three close results.

The equations to find the costs are these (for your information): Holiday Lodge, 500+12(n-25); Galaxy Inn, 17n in which n is bigger than 23; and Noble Pines Country Club, 14n + 90. Using these equations, I found that the cost for each place is 1054 for the Galaxy Inn, 958 for the Noble Pines Country Club and 944 for the Holiday Lodge. It looks like Holiday Lodge offers the lowest price for 62 people. See graph #1 and table #4 to get the full comparison of the three hotels.

The Holiday Lodge might not be the best choice though. Since it is so cheap, we could assume that the refreshments will not be good and the room might be bare and small. You might want to consider the Noble Pines Country Club because the price is in the middle and the price is closer to cheap than expensive.

Secretary of Fund raising Committee,
Jennifer

This report contains the three elements presented in the Level 3 criteria of the scoring guide on p. 89. The number of guests is estimated to be between 60 and 64. The algebraic expressions that define the cost functions are also presented correctly. The ranges of cost for between 60 and 64 guests are correctly quoted for each location, and displayed in the table of values shown here with the cheapest price for each *n* highlighted in a rectangular box. The student has also indicated that the Noble Pines C.C. is the better choice for $n < 55$, and has suggested that price may not be the only consideration. The report indicates that the dimensions of the room and the quality of the food must also be considered. In the spreadsheet exercise on page 87, this student compared all three locations for the reduced cost functions. The student response is shown below.

3. You can use your answer to question 2 to help you solve question 3.
All of the 6 elevators can take 1600 people per hour. How many people can one elevator carry in one trip?

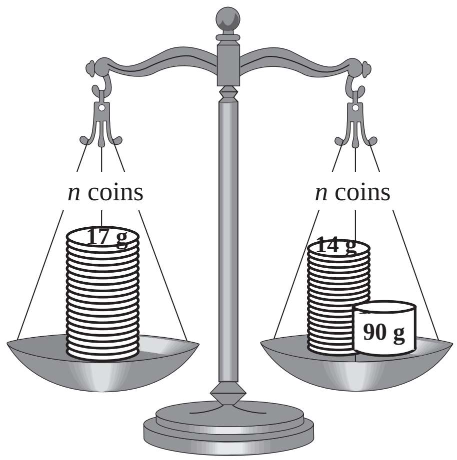
Student Spreadsheet Response ➤

out level 120 times a day, how many times would all of them together visited the look out level?
D) If the 360 restaurant roloves once ever 72 minutes, how many times would it rota in 24 ho...
D) If it's annual attendence is 1.5 mill'on sh.

Solutions to Challenges in Reasoning

Problem 1

Mr. Scrooze is a miser who has coins with masses of 14 g and 17 g. He places a stack of 17-g coins on one pan of a two-pan balance and the same number of 14-g coins on the other pan. To balance both pans, he adds a 90-g mass to the pan with the 14-g coins. How many coins are on each pan?



In the left pan

The mass of the n coins in the left pan is $n \times 17$ or $17n$ grams.

In the right pan

The mass of the n coins in the right pan is $n \times 14$ or $14n$ grams.
The mass of the 90-g mass in the right pan is 90 g.
The total mass in the right pan is $14n + 90$ grams.

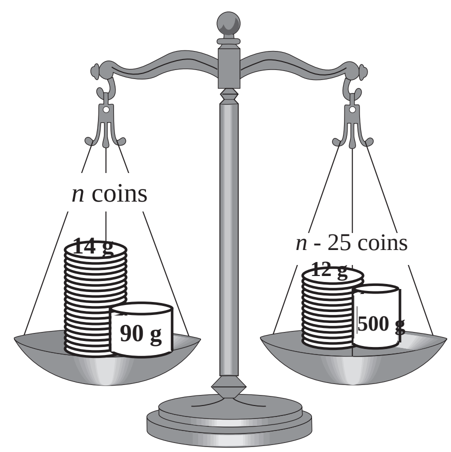
Since the scale is in balance, the mass in the left pan is equal to the total mass in the right pan. That is,

$$17n = 14n + 90.$$

Once we have this equation, we can solve it by observing that the extra $3n$ on the left side of the equation balances the extra 90 on the right side, so $n = 30$. More formally, we can subtract $14n$ from both sides of the equation to obtain $3n = 90$ and then divide both sides of this equation by 3 to obtain $n = 30$. That is, there are 30 coins of mass 12 g and 30 coins of mass 14 g.

Problem 2

In his next balancing act, Mr. Scrooze has coins with masses of 12 g and 14 g. He places all his 14-g coins and his 90-g mass on one pan of a two-pan balance. He has 25 fewer 12-g coins which he places on the other pan. To balance both pans, he adds a 500-g mass to the pan with the 12-g coins. How many coins are on each pan?



In the left pan

As in Problem 1, we find the total mass in the left pan is $14n + 90$ grams.

In the right pan

The mass of the $n - 25$ coins in the right pan is $(n - 25) \times 12$ or $12(n - 25)$ grams.
The mass of the 500-g mass in the right pan is 500 g.
The total mass in the right pan is $12(n - 25) + 500$ grams.
We can simplify this expression as follows:

$$\begin{aligned} 12(n - 25) + 500 &= 12n - 12 \times 25 + 500 \\ &= 12n + 200. \end{aligned}$$

That is, the total mass in the right pan is $12n + 200$ grams.

Since the scale is in balance, the mass in the left pan is equal to the total mass in the right pan. That is, $14n + 90 = 12n + 200$.

Subtracting $12n + 90$ from both sides of the equation yields $2n = 110$, so $n = 55$. That is, there are 55 coins of mass 14 g and 30 coins of mass 12 g.

Alternatively, we could reason that the extra 25 coins of mass 14 g have a total mass of 350 g, that combined with the 90-g mass gives the left pan $n - 25$ coins plus 440 g. Therefore the $n - 25$ coins of 14 g must have total mass 60 g more than the $n - 25$ coins of 12 g. That is, $n - 25 = 30$, so $n = 55$.

TEMPLATE

Record of Student Achievement on the Grade 7 Unit

How Many Dots in a Triangular Array with *n* Dots on Each Side?

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Communication	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Application	
Application	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Communication	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Application	
Application	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Communication	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Application	
Application	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Communication	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Concepts	
Application	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Application	
Application	
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	
Communication	

TEMPLATE

Record of Student Achievement on the Grade 8 Unit

Where Should They Hold the Fundraising Party?

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Concepts	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	
Concepts	
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Concepts	
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Concepts	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	
Concepts	
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Concepts	
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Concepts	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	
Concepts	
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Concepts	
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Concepts	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Concepts	
Concepts	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	
Concepts	
Application	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Concepts	
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Application	
Communication	

Additional Resources for Patterns & Algebra

Curcio, Francis, Barbara Nimerofsky, Rossana Perez, and Shirel Yaloz. "Exploring Patterns in Nonroutine Problems." Focus Issue of *Mathematics Teaching in the Middle School*. Vol 2, #4. Reston, VA: National Council of Teachers of Mathematics, February, 1997, pp. 262-269.

Day, Roger, and Graham A. Jones. "Building Bridges to Algebraic Thinking." Focus Issue of *Mathematics Teaching in the Middle School*. Vol 2, #4. Reston, VA: National Council of Teachers of Mathematics, February, 1997, pp. 208-212.

Fouche, Katheryn. "Algebra for Everyone: Start Early." Focus Issue of *Mathematics Teaching in the Middle School*. Vol 2, #4. Reston, VA: National Council of Teachers of Mathematics, February, 1997, pp. 226-229.

Kenney, Patricia A., Judith S. Zawojewski, and Edward A. Silver. "Marcy's Dot Pattern." Mathematics Teaching in the Middle School. Vol 3, #7. Reston, VA: National Council of Teachers of Mathematics, May, 1998, pp. 474-477.

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Phillips, Elizabeth, et al. *Patterns & Functions* from Addenda Series, Grades 5–8: Curriculum and Evaluation Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics, 1991.

Silver, Edward. "Rethinking 'Algebra for All'." *Educational Leadership,* Vol 52, #6, 1995, pp. 30-34.

van Reeuwijk, Martin, and Monica Wijers. "Students' Construction of Formulas in Context." Focus Issue of *Mathematics Teaching in the Middle School*. Vol 2, #4. Reston, VA: National Council of Teachers of Mathematics, February, 1997, pp. 230-236.

Videotape

Apostol, Tom. *The Theorem of Pythagoras: Project Mathematics!* California Institute of Technology, 1988.

Web Sites

Centre for Innovation in Math Teaching, Exeter	http://www.ex.ac.uk/cimt/
Nrich: Cambridge Centre	http://www.nrich.maths.org.uk/
Birmingham Math Links	http://www.bham.ac.uk/education/maths/links/
Algebra for Everyone Home Page	http://act.psy.cmu.edu/ACT/awpt/algebra-home.html

Free Software for Ontario Schools

The Ministry of Education and Training of Ontario purchases site licences of software for all publically funded schools in the province. This software can be obtained from the Ontario Educational Software Service (OESS) representative in your school district. To determine what is available, access this web site: **http://www.tvo.org/osapac**